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BOSTON UNIVERSITY

GRADUATE SCHOOL

Dissertation

THE MODULATION FUNCTION IN OPTICS

by

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INTRODUCTION

Originally the purpose of this investigation was to extend the concepts of Communication and Information Theory to the problems of the analysis and synthesis of optical systems. It was decided early in the investigation that before the general synthesis procedure should be undertaken, an attempt should be made to analyze several well known ideal systems in geometrical and physical optics which could be dealt with in closed analytic form. This paper presents the results of this analysis.

Since the foundation of the linear passive network analogue in optics had already been laid (refs. 3 & 5), an extended survey of the literature revealed that Schade (ref. 11) in his work in television and Marechal (ref. 17) in diffraction theory had been employing concepts, well known to communication engineers, but which appeared quite nebulous to those working in the field of optics. The key to the analysis and synthesis procedure in communication theory is the "system function", defined as the Fourier transform of the system's impulse response function. This is to be identified with Schade's "sine wave flux response function" and Marechal's "modulation function," defined as the Fourier transform of the optical system's intensity response to a point source. It is not difficult to show that such a function can be interpreted as a contrast rendition versus spatial frequency curve for sine wave targets. Thus it is possible to measure simultaneously the contrast rendition and resolution limit of an optical system by evaluating the modulation function. The problem of analytically computing this function and interpreting it was, mathematically speaking, an intriguing one. Much of the emphasis has been placed on the

analysis of several ideal optical systems in the terminology of linear passive network theory.

Section I presents the mathematical background for the subsequent use of Fourier series and integrals. While the major portion of the material can be found in the literature, it was necessary to derive the orthogonality conditions for continuous eigenvalues in order to present the methods of expanding functions into series and integrals in a unified fashion.

In Section II, the general theory of diffraction is presented and emphasis is placed on the various Fourier relations which arise. Here again, the demonstration of these Fourier transform relationships in diffraction theory date back to Michelson. However, the interpretations of these concepts in terms of network theory is stressed. The evaluation of the modulation function by means of a convolution integral over the aperture was presented by Luneberg (ref. 6) and later emphasized by Marechal in a lecture at the Boston University Physical Research Laboratories. To test the theory, several modulation functions were devised for apertures of simple geometry. Since the modulation function is defined as the Fourier transform of the point image intensity distribution, it was decided to attempt the double integration to corroborate the result of the geometrical approach and in the hope that the method would be applicable in cases where it was not possible to perform the convolution integral analytically. The scheme employed was that of integration with respect to a parameter, the results of which were indeed gratifying. It is hoped that some of the relations which were encountered will be of use in more advanced treatments in the future. The results of the double integrations and geometrical approach were in agreement with Steel

(ref. 13) who had since published his method of obtaining the modulation function for annuli by finding the response of the system to a line source and taking its one dimensional Fourier Transform. Further emphasis is placed on the modulation function as a measure of contrast rendition .

Section III is concerned with several ideal systems which are approximated in television scanning systems and geometrical optics. An extensive treatment of scanning apertures and the corresponding transfer characteristic function had been presented by Schade (ref. 11), hence it was decided to present several general criteria for the quality of a system in terms of its capability of edge preservations. In this regard, several cases were considered whereby it was possible to establish certain quality characteristics in terms of edge deterioration. A treatment of the phenomena of "spurious resolution" from the examination of the modulation function is also presented, along with an outline of the extension to random detail and the general synthesis procedure. Finally the restrictions and limitations on the general synthesis problem along with references to actually existing synthesizing operations are given.

In conclusion, the author wishes to express his gratitude to Dr. T. P. Cheatham and Dr. A. Kohlenberg for originally suggesting the investigation and for my introduction to the theory of the analysis and synthesis of a linear passive system. For their continued assistance and guidance I am indebted to Dr. D. Smith and Dr. W. Hauser. Finally for their invaluable suggestions and aid I should like to thank Mr. J. Gerdes and Mr. T. Hoban.

I. MATHEMATICAL PRELIMINARIES

A. . Eigenfunctions and Eigenvalues

The mathematical framework for the remainder of the paper is presented in one unit in order to preserve continuity in the main body of the dissertation.

Initially the eigenfunction solutions of certain partial differential equations arising in mathematical physics will be assumed. Hence, as a starting point, let us postulate, for generality, the Sturm-Liouville differential equation;

1.1
$$\frac{d}{dx}\left(P(x)\frac{d\psi}{dx}\right) + \left[q(x) + \lambda r(x)\right]\psi(x) = 0$$

Those familiar with mathematical physics will recognize this as the generalized form of the separated equation where λ is the separation constant, i. e., the eigenvalue. Corresponding to each eigenvalue λ_n , there exists an eigenfunction which satisfies equ. (1.1)

Hence we may write;

1.2
$$\frac{d}{dx}\left(P(x)\frac{d\psi_n}{dx}\right) + \left[q(x) + \lambda_n r(x)\right]\psi_n(x) = 0$$

where λ is real

and similarly;

1.3
$$\frac{d}{dx}\left(P(x)\frac{d\psi_m^*}{dx}\right) + \left[q(x) + \lambda_m r(x)\right]\psi_m^*(x) = 0$$

the starred notation implying the complex conjugate.

If we now multiply (1.2) by ψ_m^* and (1.3) by ψ_n , subtract and integrate over the region for which $\psi(x)$ is defined, we have;

1.4 $$\int_a^b \left\{ \psi_m^* \left[\frac{d}{dx}\left(P(x)\frac{d\psi_n}{dx}\right)\right] - \psi_n\left[\frac{d}{dx}\left(P(x)\frac{d\psi_m^*}{dx}\right)\right]\right\} dx + (\lambda_m - \lambda_n)\int_a^b r(x)\,\psi_n\,\psi_m^*\, dx = 0$$

At this point it must be remarked that this differential equation is concerned with boundary value problems, the physical constraints of which demand that ψ or $\frac{\partial \psi}{\partial x}$ vanish at the boundaries, or perhaps $P(x)$.

With this in mind, let us now integrate the first integral of (1.4) by parts;

1.5 $$\int_a^b \psi_m^*\left[\frac{d}{dx}\left(P(x)\frac{d\psi_n}{dx}\right)\right]dx = \psi_m^*\, P(x)\frac{d\psi_n}{dx}\Bigg|_a^b - \int_a^b P(x)\frac{d\psi_m^*}{dx}\,\frac{d\psi_n}{dx}\,dx$$

and similarly

1.6 $$\int_a^b \psi_n\left[\frac{d}{dx}\left(P(x)\frac{d\psi_m^*}{dx}\right)\right]dx = \psi_n\, P(x)\frac{d\psi_m^*}{dx}\Bigg|_a^b - \int_a^b P(x)\frac{d\psi_n}{dx}\,\frac{d\psi_m^*}{dx}\,dx$$

Now from the conditions cited above on ψ and $\frac{\partial \psi}{dx}$ at the boundaries and by substituting (1.5) and (1.6) in (1.4), we have;

1.7 $$(\lambda_m - \lambda_n)\int_a^b r(x)\,\psi_n\,\psi_m^*\,dx = 0$$

If we assume non-degenerate eigenvalues, i.e., $\lambda_n \neq \lambda_m$, we have the extremely important relationship;

1.8 $$\int_a^b r(x)\,\psi_n\,\psi_m^*\,dx = C^2\,\delta_{nm}$$

where δ_{nm} is the standard "Kronecker delta" having the interpretation

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Thus the ψ's form an orthogonal set and $r(x)$ is called the weighting function. In order to normalize this set of functions, it is necessary

merely to define a new set such that;

$$\psi_n' = \frac{\psi_n}{c} \qquad \psi_m^{*'} = \frac{\psi_m^*}{c}$$

then;

1.9 $$\int_a^b \psi_n' \, \psi_m^{*'} \, r(x) \, dx = \delta_{mn}$$

However, in what is to follow we shall assume that the normalization constant has been absorbed into the eigenfunctions in order to dispense with the primed notation. For a more comprehensive treatment on the completeness of eigenfunctions, degeneracy and Sturm-Liouville theory in general, the reader should consult ref. #8 and the additional references cited therein.

As an aside, we may mention that two vectors $\vec{A}$ and $\vec{B}$ are said to be "orthogonal" if their scalar product vanishes.

1.10 $$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i = 0$$

In like manner, in n-dimensional space, the orthogonality condition becomes;

1.11 $$\sum_{i=1}^{n} A_i B_i = 0$$

If we now consider an infinite dimensional space in which the components A_i and B_i become continously distributed, then i becomes a continuous variable instead of a denumerable index and we have for the scalar product;

1.12 $$\int_a^b A(x) \, B(x) \, dx = 0$$

This definition of orthogonality, of course, assumes a specific range of integration to be meaningful.

As for the property of completeness, the reader again is referred to refs. #8 and #9 wherein this property is established from a variational standpoint. Suffice to say at present, a set of functions is complete if an arbitrary function satisfying the same boundary conditions as the functions of the set can be expanded in a series with constant coefficients, i. e.,

1.13 $$f(x) = \sum_{n=1}^{\infty} C_n \psi_n(x)$$

From the orthogonality property previously established, one can evaluate the coefficients by multiplying both sides of (1.13) by $\psi_m^* r(x)$ and integrating;

1.14 $$\int_a^b f(x)\,\psi_m^*\, r(x)\,dx = \sum_{n=1}^{\infty} C_n \int_a^b \psi_m^* \psi_n\, r(x)\,dx = \sum_{n=1}^{\infty} C_n\, \delta_{mn}$$

or

1.15 $$C_m = \int_a^b f(x)\,\psi_m^*\, r(x)\,dx$$

In this way we have a series expansion for $f(x)$ in terms of eigenfunctions and constant coefficients which can be readily evaluated. The constant terms are called "Fourier coefficients".

Thus far we have dealt with eigenfunctions defined over a finite region of integration, the eigenvalues taking on discrete values. Further on we shall show that over an infinite domain for the eigenfunctions in which we are interested, the eigenvalues become continuous and the normalization

and orthogonality relation is given by

1.16 $$\int_{-\infty}^{\infty} \psi(\lambda, x)\, \psi^*(\lambda', x)\, r(x)\, dx = \delta(\lambda - \lambda')$$

where $\delta(\lambda - \lambda')$ is the "Dirac delta function" which enjoys the unusual property of being zero everywhere except at the point, $\lambda = \lambda'$ and there being infinite, but in such a way that

1.17 $$\int_{-\infty}^{\infty} g(x)\, \delta(x-a)\, dx = g(a)$$

Other properties of this "function" are well known and can readily be found in the literature.*

In a completely analogous fashion,therefore, it is possible to express a function in an integral form, in terms of eigenfunctions which possess continuous eigenvalues.

1.18 $$f(x) = \int_{-\infty}^{\infty} C(\lambda)\, \psi(\lambda, x)\, d\lambda$$

Now multiplying both sides by $\psi^*(\lambda', x)\, r(x)$ and integrating, we have:

1.19 $$\int_{-\infty}^{\infty} f(x)\, \psi^*(\lambda', x)\, r(x)\, dx = \int_{-\infty}^{\infty} C(\lambda)\, d\lambda \int_{-\infty}^{\infty} \psi(\lambda, x)\, \psi^*(\lambda', x) \overset{r(x)}{\wedge} dx = \int_{-\infty}^{\infty} \delta(\lambda - \lambda')\, C(\lambda)\, d\lambda$$

whence

1.20 $$C(\lambda') = \int_{-\infty}^{\infty} f(x)\, \psi^*(\lambda', x)\, r(x)\, dx$$

Hence we see that the methods for discrete and continuous eigenvalues are completely analogous once the normalization and orthogonality properties

---

*cf. for example Schiff's "Quantum Mechanics" rf. #12

of the eigenfunctions have been established.

B. Fourier Series and Integral

Let us now consider the one-dimensional scalar Helmholtz equation which arises, for example, in the problem of a vibrating string.

1.21
$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

We see that this is a special form of the Sturm-Liouville differential equation (1.1) with: $p(x) = 1, \quad q(x) = 0, \quad r(x) = 1; \quad \lambda = k^2$

The solutions to this equation are the eigenfunctions

1.22
$$\psi \sim e^{\pm ikx} \sim \begin{cases} \cos kx \\ \sin kx \end{cases}$$

And the physical restrictions demand, for example $[\psi(0) = \psi(\ell) = 0]$, that the eigenvalue take the form

1.23
$$k_n = \frac{2\pi n}{\ell} \qquad n = 0, 1, 2, \cdots$$

thus the eigenfunctions for this equation are

1.24
$$\psi_n = e^{i\frac{2\pi n x}{\ell}} \qquad n = -\infty \cdots -2, -1, 0, 1, 2, \cdots \infty$$

and further we see that

1.25
$$\int_{-\ell/2}^{\ell/2} \psi_n \psi_m^* \, dx = \int_{-\ell/2}^{\ell/2} e^{i\frac{2\pi(n-m)x}{\ell}} \, dx = \ell\,\delta_{mn}$$

let us now define $\omega = \frac{2\pi}{\ell}$ and our eigenfunctions as

1.26
$$\psi_n = \frac{e^{in\omega x}}{\sqrt{\ell}}$$

with this notation we can now expand a periodic function in the interval $-\frac{l}{2} \le x \le \frac{l}{2}$ in terms of these eigenfunctions.

1.27
$$f(x) = \sum_{n=-\infty}^{\infty} \frac{C_n e^{in\omega x}}{\sqrt{l}}$$

To determine the C_n's we multiply both sides by $e^{-im\omega x}$ and integrate:

1.28
$$\int_{-l/2}^{l/2} f(x)\frac{e^{-im\omega x}}{\sqrt{l}}\,dx = \sum_{n=-\infty}^{\infty} C_n \int_{-l/2}^{l/2} \frac{e^{i(n-m)\omega x}}{l}\,dx = \sum_{n=-\infty}^{\infty} C_n\,\delta_{nm}$$

or

1.29
$$C_m = \frac{1}{\sqrt{l}}\int_{-l/2}^{l/2} f(x)\, e^{-im\omega x}\,dx$$

Now for non-periodic functions we let l approach infinity, and if $\frac{C_m}{\sqrt{\omega_1}}$ remains finite: that is, since $\omega_n = n\omega_1$ and $l = \frac{2\pi}{\omega_1}$ 1.29 becomes

1.30
$$\frac{C_m}{\sqrt{\omega_1}} = \frac{1}{\sqrt{2\pi}}\int_{-\pi/\omega_1}^{\pi/\omega_1} f(x)\, e^{-i\omega_n x}\,dx$$

Now as $l \to \infty$; $\omega_1 \to 0$ and the relative amplitudes of the spectrum come closer together and eventually become continuous. Despite the fact that C_m also approaches zero, it is reasonable to expect that $\frac{C_m}{\sqrt{\omega_1}}$ remains finite. With this in mind, let us rewrite (1.27) by multiplying the numerator and denominator of the right hand member by $\omega_1 = \Delta\omega$

1.31
$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\left(\frac{C_n}{\sqrt{\omega_1}}\right) e^{i\omega_n x}\,\Delta\omega$$

Now taking the limit of equations(1.30) and (1.31) in the following manner:

$$l \to \infty \qquad \omega_n \to \omega$$
$$\omega_1 \to 0 \qquad \Delta\omega \to d\omega$$

and let us now define:

1.32
$$g(\omega) = \frac{C_m}{\sqrt{\omega_1}}$$

whence, in the limit

1.33
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega x}\, d\omega$$

where

1.34
$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

It is precisely these equations which we now define as Fourier transform pairs. Here $f(x)$ represents a non-periodic function which obeys certain general restrictions which shall be mentioned further on, and $g(\omega)$ represents the continuous spectral density which makes up $f(x)$, the actual complex harmonic amplitudes being given by $g(\omega)\, d\omega$.

We shall now prove the normalization and orthogonality properties of the eigenfunction $\frac{e^{i\omega x}}{\sqrt{2\pi}}$ as stated in equation 1.6 namely:

1.35
$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\sqrt{2\pi}} \frac{e^{-i\omega' x}}{\sqrt{2\pi}}\, dx = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^{R} e^{i(\omega - \omega')x}\, dx \quad *$$
$$= \frac{1}{\pi} \lim_{R \to \infty} \frac{\sin(\omega - \omega')R}{\omega - \omega'}$$

hence

1.36
$$\int_{-\infty}^{\infty} \frac{e^{i(\omega - \omega')x}}{2\pi}\, dx = \delta(\omega - \omega')$$

At this point let us now summarize the procedure of expanding a function into its sine wave components.

---

* cf. Ref. 12 Wherein Schiff defines $\delta(x) = \lim_{\beta \to \infty} \frac{1}{\pi} \frac{\sin \beta x}{x}$

a) Periodic function $\omega = \frac{2\pi}{P}$

1.37
$$f(x) = \frac{1}{\sqrt{P}} \sum_{n=-\infty}^{\infty} C_n e^{in\omega x}$$

Multiplying through by $e^{-im\omega x}/\sqrt{P}$ and integrating we have:

1.38
$$\frac{1}{\sqrt{P}} \int_{-P/2}^{P/2} f(x) e^{-im\omega x}\, dx = \sum_{n=-\infty}^{\infty} C_n \int_{-P/2}^{P/2} \frac{e^{i(n-m)\omega x}}{P}\, dx = \sum_{n=-\infty}^{\infty} C_n \delta_{mn}$$

whence

1.39
$$C_m = \frac{1}{\sqrt{P}} \int_{-P/2}^{P/2} f(x) e^{-im\omega x}\, dx$$

Hence the amplitudes of the sine wave components are discrete and their magnitude is given by (1.39)

b) Non-Periodic Functions $(-\infty < x < \infty)$

1.40
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega x}\, d\omega$$

In a completely analogous fashion, multiply through by $e^{-i\omega' x}/\sqrt{2\pi}$ and integrate

1.41
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega' x}\, dx = \int_{-\infty}^{\infty} g(\omega)\, d\omega \int_{-\infty}^{\infty} \frac{e^{i(\omega-\omega')x}}{2\pi}\, dx = \int_{-\infty}^{\infty} \delta(\omega-\omega')\, g(\omega)\, d\omega$$

whence

1.42
$$g(\omega') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega' x}\, dx$$

Here the amplitudes of the sine wave components are continuous and their magnitude is given by $g(\omega)\, d\omega$.

C. Multi-dimensional Fourier representation *

The extension of this procedure to n-dimensions will not be presented here. However, the method is entirely analogous to the one-dimensional case and the results will be demonstrated.

In the case of discrete eigenvalues the orthonormal property becomes

1.43 $$\int\int_{-\frac{P_{x_i}}{2}}^{\frac{P_{x_i}}{2}}\cdots\int \frac{e^{i\omega\sum_{i=1}^{n}(m_i-m_i')x_i}}{(P)^n}\,dx_1\,dx_2\cdots dx_n = \delta_{m_1m_1'}\,\delta_{m_2m_2'}\cdots\delta_{m_nm_n'}$$

Thus we can expand $f(x_1, x_2, \cdots x_n)$ in the following series

1.44 $$f(x_1, x_2, \cdots x_n) = \left(\frac{1}{\sqrt{P}}\right)^n \sum_{m_1=-\infty}^{\infty}\sum\cdots\sum_{m_n=-\infty}^{\infty} C_{m_1, m_2, \cdots m_n}\, e^{i\omega\sum_{j=1}^{n} m_j x_j}$$

Now multiplying both sides by $e^{-i\omega\sum m_j' x_j}\Big/(\sqrt{P})^n$ and integrating we have:

1.45 $$\left(\frac{1}{\sqrt{P}}\right)^n\int\int_{-\frac{P_{x_i}}{2}}^{\frac{P_{x_i}}{2}}\cdots\int f(x_1, x_2, \cdots x_n)\, e^{-i\omega\sum m_j' x_j}\,dx_1\,dx_2\cdots dx_n = \sum_{m_1=-\infty}^{\infty}\sum\cdots\sum_{m_n=-\infty}^{\infty} C_{m_1\cdots m_n}\delta_{m_1m_1'}\cdots\delta_{m_nm_n'}$$

whence:

1.46 $$C_{m_1', m_2', \cdots m_n'} = \left(\frac{1}{\sqrt{P}}\right)^n\int\int\cdots\int f(x_1, x_2, \cdots x_n)\, e^{-i\omega\sum_{j=1}^{n} m_j' x_j}\,dx_1\,dx_2\cdots dx_n$$

In like manner we can extend the procedure for transient functions to n-dimensions, namely:

1.47 $$f(x_1, x_2, \cdots x_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n\int\int_{-\infty}^{\infty}\cdots\int g(\omega_1, \omega_2, \cdots \omega_n)\, e^{i\vec{\omega}\cdot\vec{X}}\,d\omega_1\,d\omega_2\cdots d\omega_n$$

where $\vec{\omega}\cdot\vec{X} = \sum_{j=1}^{n}\omega_j x_j$

Again making use of the orthonormal property

1.48 $$\int\int\cdots\int\frac{e^{i\sum_{j=1}^{n}(\omega_j-\omega_j')x_j}}{(\sqrt{2\pi})^n}\,dx_1\,dx_2\cdots dx_n = \delta(\omega_1-\omega_1')\,\delta(\omega_2-\omega_2')\cdots\delta(\omega_n-\omega_n')$$

we can evaluate $g(\omega_1, \omega_2, \cdots \omega_n)$

---

* cf. ref. #8

i. e.,

1.49 $$g(\omega_1', \omega_2', \cdots \omega_n') = \left(\frac{1}{\sqrt{2\pi}}\right)^n \iint_{-\infty}^{\infty} \cdots \int f(x_1, x_2, \cdots x_n)\, e^{-i\vec{\omega}\cdot\vec{X}}\, dx_1\, dx_2 \cdots dx_n$$

One case of particular interest is the two-dimensional problem with circular symmetry. We can approach this problem either from the two dimensional Fourier representation in cartesian coordinates and transform to polar coordinates, or starting with scalar Helmholtz equation in polar coordinates. Actually we shall make use of both schemes, but to be consistent we shall introduce the orthogonal functions and their properties from the scalar Helmholtz equation as a special case of the Sturm-Liouville differential equation.

Namely

1.50 $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + k^2\psi = 0$$

Employing the method of separation of variables we find,

1.51 $$\frac{d^2\Theta}{d\theta^2} = -m^2\Theta \;;\quad \Theta = Ae^{im\theta} \qquad \begin{array}{c} m = 0, 1, 2, \cdots \\ 0 \le \theta \le 2\pi \end{array}$$

and 1.52 $$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(k^2 r - \frac{m^2}{r}\right)R = 0 \;;\qquad R = BJ_m(kr)$$

where $\psi = R(r)\,\Theta(\theta)$

Equation 1.52 being immediately recognized as Bessels equation. We have implicitly rejected the Bessel function of the second kind as a solution to 1.52 since it approaches infinity as r approaches zero. What is of interest however, is that the eigenfunctions for two dimensions

in polar coordinates are of the form

1.53 $$\psi \sim e^{im\theta} J_m(kr)$$

Again, from the physical constraints imposed on most problems wherein this function arises, e. g. $\psi(a) = 0$, k must take on some characteristic or eigenvalue which we shall denote by

1.54 $$k = \frac{\nu_{m\ell}}{a} = \omega_{m\ell}$$

Noting that the weighting function from (1.52) is r we have for the orthogonality condition

1.55 $$\int_0^a\int_0^{2\pi} J_m(\omega_{m\ell} r) e^{im\theta} \cdot J_n(\omega_{n\ell'} r) e^{-in\theta} r\, dr\, d\theta = 2\pi \delta_{mn} \int_0^a J_m(\omega_{m\ell} r) J_n(\omega_{n\ell'} r) r\, dr$$

$$= 2\pi \int_0^a J_m(\omega_{m\ell} r) J_m(\omega_{m\ell'} r)\, r\, dr$$

$$= \frac{2\pi a^2}{2} J^2_{m+1}(\omega_{m\ell} a)\, \delta_{\ell\ell'}$$

$$= \pi a^2 J^2_{m+1}(\omega_{m\ell} a)$$

Thus the eigenfunction is given by

1.56 $$\psi_{m\ell} = \frac{J_m(\omega_{m\ell} r)\, e^{im\theta}}{\sqrt{\pi}\, a\, J_{m+1}(\omega_{m\ell} a)}$$

Let us now consider the expansion of any function $f(r,\theta)$ in terms of these eigenfunctions

1.57 $$f(r,\theta) = \sum_m \sum_\ell \frac{C_{m\ell}}{a\sqrt{\pi}\, J_{m+1}(\omega_{m\ell} a)} e^{im\theta} J_m(\omega_{m\ell} r)$$

in order to evaluate the $C_{m\ell}$'s we multiply through by $\dfrac{e^{-in\theta} J_n(\omega_{n\ell'} r)}{a\sqrt{\pi}\, J_{n+1}(\omega_{n\ell'} a)}$ and integrate:

1.58 $$\frac{1}{a\sqrt{\pi}\,J_{n+1}(\omega_{nl'}a)}\int_0^a\int_0^{2\pi} f(r,\theta)\,e^{-in\theta}\,J_n(\omega_{nl'}r)\,r\,dr\,d\theta = \sum_m \sum_l C_{ml}\,\delta_{mn}\,\delta_{ll'}$$

or

1.59 $$C_{nl'} = \frac{1}{a\sqrt{\pi}\,J_{n+1}(\omega_{nl'}a)}\int_0^a\int_0^{2\pi} f(r,\theta)\,e^{-in\theta}\,J_n(\omega_{nl'}r)\,r\,dr\,d\theta$$

It is seen that the method is entirely analogous to the one-dimensional Fourier series.

Again it will be postulated that as $a \to \infty$ the ω_{ml}'s come closer together and in the limit become continuous. The result of the limiting process is the Fourier-Bessel transform, and for the special case where the function may be represented as $f(r,\theta) = f(r)\,e^{im\theta}$

1.60 $$f(r) = \int_0^\infty g(\omega)\,J_m(\omega r)\,\omega\,d\omega$$

and

1.61 $$g(\omega) = \int_0^\infty f(r)\,J_m(\omega r)\,r\,dr$$

The application of this pair is now obvious from what has proceeded once we establish the orthogonality property for the Bessel function over an infinite domain. That is, we must prove that

1.62 $$\int_0^\infty J_m(\omega r)\,J_m(\omega' r)\,r\,dr = \frac{1}{\omega}\,\delta(\omega-\omega')$$

To do this we make use of the following scheme;

let

1.63 $$\int_0^\infty J_m(\omega r)\,J_m(\omega' r)\,r\,dr = \lim_{R\to\infty}\int_0^R J_m(\omega r)\,J_m(\omega' r)\,r\,dr$$

whence:

1.64* $$\lim_{R\to\infty}\int_0^R J_m(\omega r)J_m(\omega' r)\,r\,dr = \lim_{R\to\infty} R\left[\frac{\omega J_m(\omega' R)J'(\omega R)-\omega' J_m(\omega R)J_m'(\omega' R)}{\omega^2-\omega'^2}\right]$$

making use of the recurrence relation

** 1.65 $$2J_n' = J_{n-1} - J_{n+1}$$

1.64 becomes

1.66 $$\lim_{R\to\infty}\frac{R}{2}\left[\omega J_m(\omega' R)J_{m-1}(\omega R) - \omega' J_m(\omega R)J_{m-1}(\omega' R) + \omega' J_n(\omega R)J_{m+1}(\omega' R) - \omega J_m(\omega' R)J_{m+1}(\omega R)\right]$$

Employing the asymptotic form for the Bessel functions

1.67 $$\lim_{\beta\to\infty} J_m(\beta) = \sqrt{\frac{2}{\pi\beta}}\,\cos\left(\beta - \frac{\pi}{4} - \frac{m\pi}{2}\right)$$

and letting

1.68 $$\phi_1 = \omega R - \frac{\pi}{4}(2m+1)$$

$$\phi_2 = \omega' R - \frac{\pi}{4}(2m+1)$$

1.66 becomes

1.69 $$\lim_{R\to\infty}\frac{2R}{\pi\sqrt{\omega\omega'}\,R}\left[\frac{\omega'\cos\phi_1\sin\phi_2 - \omega\cos\phi_2\sin\phi_1}{\omega^2-\omega'^2}\right] = \lim_{R\to\infty}\frac{1}{\pi\sqrt{\omega\omega'}}\left[\frac{(\omega-\omega')\sin(\phi_1+\phi_2)-(\omega+\omega')\sin(\phi_1-\phi_2)}{\omega^2-\omega'^2}\right]$$

or

1.70 $$\lim_{R\to\infty}\frac{1}{\pi\sqrt{\omega\omega'}}\left[\frac{\sin(\omega-\omega')R}{\omega-\omega'} + \frac{\sin\left\{(\omega+\omega')R-(2m+1)\frac{\pi}{2}\right\}}{\omega+\omega'}\right] = \lim_{R\to\infty}\frac{1}{\pi\sqrt{\omega\omega'}}\left[\frac{\sin(\omega-\omega')R}{\omega-\omega'} + \frac{(-1)^{m+1}\cos(\omega+\omega')R}{\omega+\omega'}\right]$$

---

* From the recurrence relationships for Bessel Functions;

** $J' = \frac{dJ}{dr}$

But recalling that Schiff (ref. 12) defines the "Dirac delta function " as

1.71 $$\lim_{\alpha \to \infty} \frac{1}{\pi} \frac{\sin \alpha x}{x} = \delta(x)$$

and noting that the second term in 1.70 is an odd function which will undergo an infinite number of oscillations that will, in the limit, average out to zero, * we have

1.72 $$\int_0^\infty J_m(\omega r) J_m(\omega' r)\, r\, dr = \frac{1}{\omega} \delta(\omega - \omega')$$

which is what we set out to prove.

Let us now illustrate the use of the orthogonality property for the Fourier-Bessel transforms.

1.73 $$f(r) = \int_0^\infty g(\omega) J_m(\omega r)\, \omega\, d\omega$$

Now multiplying through by $J_m(\omega' r)\, r\, dr$ and integrating

1.74 $$\int_0^\infty f(r) J_m(\omega' r)\, r\, dr = \int_0^\infty g(\omega)\, d\omega \int_0^\infty J_m(\omega r) J_m(\omega' r)\, r\, dr$$

$$= \int_0^\infty \frac{1}{\omega} \delta(\omega - \omega')\, g(\omega)\, \omega\, d\omega$$

or

1.75 $$g(\omega') = \int_0^\infty f(r) J_m(\omega' r)\, r\, dr$$

which verifies equation 1.61

It should be noted here that this transform is applicable only where $F(r,\theta)$ is a single valued function of θ so that it can be represented as** $f(r)\, e^{im\theta}$

---

* cf. Morse and Feshbach ref. #9 ; ** cf. ref. #14

It will be of interest to arrive at this same result by means of the two dimensional Fourier integral in cartesian coordinates. Since this has essentially been done in the references already cited, let us make use of the rather simple case for which $F(r)$ is not a function of θ

In this case $m = 0$ and we have from 1.47 in two dimensions

1.76
$$f(x, y) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} g(\omega_x, \omega_y)\, e^{i \vec{\omega} \cdot \vec{r}}\, d\omega_x\, d\omega_y$$

and

1.77
$$g(\omega_x, \omega_y) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} f(x, y)\, e^{-i \vec{\omega} \cdot \vec{r}}\, dx\, dy$$

now making use of the circular symmetry, i.e. $f(x, y) = f(r)$

1.78
$$g(\omega) = \frac{1}{2\pi} \int_0^{\infty}\int_0^{2\pi} f(r)\, e^{-i\omega r \cos(\theta - \alpha)}\, r\, dr\, d\theta$$

$$= \frac{1}{\pi} \int_0^{\infty} f(r)\, r\, dr \int_0^{\pi} \cos[\omega r \cos(\theta - \alpha)]\, d\theta$$

but

1.79
$$J_0(\beta) = \frac{1}{\pi} \int_0^{\pi} \cos(\beta \cos\phi)\, d\phi$$

hence

1.80
$$g(\omega) = \int_0^{\infty} f(r)\, J_0(\omega r)\, r\, dr$$

and from 1.72

1.81
$$f(r) = \int_0^{\infty} g(\omega)\, J_0(\omega r)\, \omega\, d\omega$$

which of course agrees with equations 1.60 and 1.61 for $m = 0$

The reason for establishing these relationships will become obvious when they are applied to problems possessing circular symmetry.

D. Conditions and restrictions on the use of Fourier Series and Integrals.

While a detailed and rigorous treatment concerning the Dirichlet conditions is not within the province of this discussion, the reader being referred to the standard texts in this field, such as ref. 1 , we nevertheless shall state the provisions for expanding a function in a Fourier series or integral.

a) Fourier series

Over one period the function must

1) possess single valuedness
2) remain finite
3) have a finite number of discontinuities
4) have a finite number of maxima and minima

b) Fourier Integral

1) $f(x)$ must be single valued in the range $-\infty < x < \infty$. However, it may have a finite number of finite discontinuities.
2) at a point of discontinuity x_0, the function will be given by the mean value

$$f(x_0) = \frac{1}{2}\left[f(x_0+0) + f(x_0-0)\right]$$

3) the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ must exist.

A word of caution is in order at this point with regard to the interpretation of ω as defined in equations 1.54 and 1.23. The reason for associating ω with

the wave number will become obvious when this material is applied to certain optical systems. In most cases, $f(x)$ will be an intensity distribution as a function of the space coordinates, hence ω will be employed as representative of a spatial frequency $[\text{rad.}]\,[L]^{-1}$ -not temporal frequency $[\text{rad.}]\ [T]^{-1}$

E. Linear Systems

While the major portion of this material can be found in references 2, 3, and 5, an outline of the more important concepts is necessary at this point to maintain continuity of thought throughout the dissertation. The basic concepts will be presented for one-dimensional systems tacitly assuming that the method is extendable to two or n-dimensions as shown in references (3) and (5) respectively.

The conditions for linearity demand that

1.82

a) if $f_{in}(x) \longrightarrow f_{out}(x')$ where $\longrightarrow$ is to be interpreted as "gives"

then $\alpha f_{in}(x) \rightarrow \alpha f_{out}(x')$ where α is an arbitrary constant

b) if $g_{in}(x) \longrightarrow g_{out}(x')$

then $f_{in}(x) + g_{in}(x) \longrightarrow f_{out}(x') + g_{out}(x')$

c) if $f_{in}(x) \longrightarrow f_{out}(x')$

then $f_{in}(x + x_0) \longrightarrow f_{out}(x' + x_0')$

Let us now imagine a physical system which possesses these properties of linearity. Then if we use a unit impulse (Dirac delta "function" previously

introduced) as an input, the system can be said to operate on this unit impulse to give some output $h(x)$.

i. e.

1.83 $$\delta_{in}(x) \rightarrow h_{out}(x')$$

In the electrical case h is the voltage vs. time response of the system to an impulse function, and in the optical case will be the intensity vs. space response of the lens to a point source, to cite but two illustrations.

From the linearity conditions, if our input is now made up of a series of discrete points of varying magnitude f_{in_j} we have

1. 84 $$f_{out}(x) = \sum_{j=-n}^{n} h(x-x_0)\, f_{in_j}(x_0)$$

and in the limit when the input is a continuous function 1. 84 becomes

1.85 $$f_{out}(x) = \int_{-\infty}^{\infty} h(x-x_0)\, f_{in}(x_0)\, dx_0$$ cf. Fig. 1-1

It may be recognized from 1.85 that h plays the role of a Green's function or response function that arises in many of the advanced problems in Mathematical Physics. From 1.85 we see also that in theory at least, one can evaluate the output of a linear physical system by integration, knowing the response function. Indeed, if we had n such systems coupled together, each having its own response function we would have to perform n such integrations in succession to evaluate the final output. In practice, of course, this proves quite tedious, if not impossible. However, the lengthy discourse on the applicability of Fourier transforms now will

become obvious if we take the Fourier Transform of both sides of equation (1.85)

1.86 $$F_{out}(\omega) = H(\omega)\, F_{in}(\omega)$$

where $F_{out}(\omega)$ and $F_{in}(\omega)$ represent the spectrum of the output and input respectively. If we now consider our n-coupled systems each having its own system function, $H_n(\omega)$; the evaluation of the final output spectrum reduces to n successive multiplication processes. It is precisely this tremendous simplification which makes the Fourier method such a powerful tool.

It is important at this point to elaborate on the significance of equation 1.86. A convenient way of interpreting it, is to think of $H(\omega)$, the system function or modulation function as operating, as it were, on the spectrum of the input $F_{in}(\omega)$. That is, in a picturesque fashion, the physical system "sees" only sine waves and acts on them separately. Hence, knowing $H(\omega)$, and plotting it together with the sine wave components (spectrum) of the input, performing the appropriate multiplication, the result will be the output spectrum, from which of course the actual output could be constructed. As previously stated, except for a factor of $\sqrt{2\pi}$, $H(\omega)$ is the exact Fourier transform of the system's response to an impulse function. Applications illustrating this point will be discussed more fully later in this paper.

* Thus a knowledge of the modulation function affords a quantitive measure of the physical system's performance.* In the next section we shall analyze

---

* N. B. The term system function $H(\omega)$ arises in electrical engineering. Recently the same function has been termed modulation function $M(\omega)$ in

several optical systems in terms of their modulation function and in fact emphasize the interpretation of these functions. Later a method will be described wherein one purposely alters the modulation function in order to demand given response characteristics of the system.

Again, for more extensive treatments of this procedure, the reader is referred to the standard works in Linear Network Theory and refs. (2), (3) and (5). An outline of the extension to random distributions in the input will be given in a later section of this paper.

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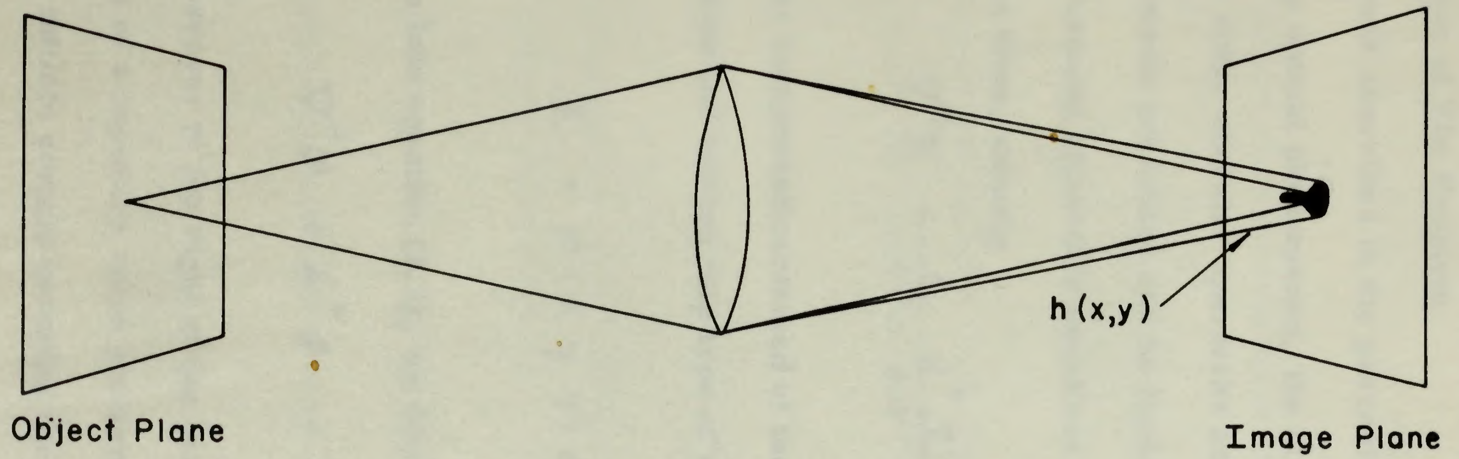
optics. It is to be noted that the two are equivalent and both will be used extensively. i. e. $M(\omega) \equiv H(\omega)$


h (x,y)
Object Plane
Image Plane

$$f_{\text{image}}(x,y) = \iint_{-\infty}^{\infty} h(x-x', y-y')\, f_{\text{obj.}}(x', y')\, dx'\, dy'$$

$$F_{\text{image}}(\omega_x,\omega_y) = M(\omega_x,\omega_y)\, F_{\text{obj.}}(\omega_x,\omega_y)$$

Fig. 1-1

II. Application of Fourier Methods to the Diffraction of Plane Waves by Apertures

A. Formulation of The Problem

The methods described in the preceding section will now be applied to the well known optical phenomenon, the diffraction of a monochromatic plane wave by slits, circular apertures and annuli. The source for much of the material herein presented can be found in refs. (13) and (17).

We shall assume Maxwell's equations and start with the wave equation derivable from them, namely

2.1 $$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0$$

where Φ is any component(cartesian) of the electromagnetic field quantities. The usual scheme for solving this type of equation is to assume a solution of the form

2.2 $$\Phi = \phi(x, y, z)\, e^{i\omega t}$$

By substitution into equation (2.1), we obtain the scalar Helmholtz equation.

2.3 $$\nabla^2 \phi + k^2 \phi = 0$$

Thus for an observer to the right of the screen (cf. Fig. 2-1) the problem reduces to that of a boundary value problem with the value of ϕ and $\frac{\partial \phi}{\partial n}$ known over an infinite domain bounding the right hand side of the screen. The method to be employed here will be that of the Green's function as illustrated in fig. 2-2. Let us define $G(r/r')$ as the response at $P(x', y', z')$

due to a point source at $P(x, y, z)$. In spherical coordinates, therefore, the Green's function satisfies the following differential equation.

2.4
$$\nabla^2 G + k^2 G = -4\pi\delta(\vec{r} - \vec{r}')$$

It can be shown* that the solution to this equation is a spherical wave emanating from $P(x, y, z)$.

2.5
$$G = \frac{e^{-ik\vec{R}'}}{\vec{R}'} \qquad \text{where } \vec{R}' = \vec{r} - \vec{r}'$$

Let us now subtract equation (2.4) multiplied by ϕ from equation 2.3 multiplied by G and integrate over the volume to the right of the screen, the result:

2.6
$$4\pi\iiint\phi(x,y,z)\delta(x-x')\delta(y-y')\delta(z-z')\,dx\,dy\,dz = \iiint(G\nabla^2\phi - \phi\nabla^2 G)\,dx\,dy\,dz$$

From the properties of the "delta function" previously established, and making use of Green's theorem**, (2.6) becomes

2.7
$$\phi(x', y', z') = \frac{1}{4\pi}\iint\left(G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right)\cdot\hat{n}\,da$$

where the integration is carried out over the area bounding the volume to the right of the screen.

At this point we shall consider P'' as a point source emitting a monochromatic spherical wave.

Hence

2.8
$$\phi = \frac{Ae^{-ikR''}}{R''}$$

---

* Morse & Feshbach, ref. #9; ** Green's Theorem

$$\iiint(A\nabla^2 B - B\nabla^2 A)\,d\tau = \iint(A\,\mathrm{grad}\,B - B\,\mathrm{grad}\,A)\cdot\hat{n}\,da$$

We thus arrive at the Kirchhoff-Huygens formulation for the diffraction problem. It is interesting to note that equation (2.7) is a neat verification of Huygens principle; since G represents the disturbance at $P'(x', y', z')$ due to a point source at $P(x, y, z)$, we have merely added up all of the contributions in the wave front ϕ in the form of an integral. Hence each point acts as a source, the net contribution being given by (2.7).

Further, since

2.9 $$\frac{\partial}{\partial n}\left(\frac{e^{-ikR}}{R}\right) = \frac{\partial}{\partial R}\left(\frac{e^{-ikR}}{R}\right)\frac{\partial R}{\partial n} = -\left(\frac{1}{R} + ik\right)\frac{e^{-ikR}}{R}\cos(\hat{n}, R)$$

(2.7) becomes

2.10 $$\phi(x', y', z') = \frac{A}{4\pi}\iint \frac{e^{-ik(R'+R'')}}{R'R''}\left[\left(\frac{1}{R} + ik\right)\cos(\hat{n}, R') - \left(\frac{1}{R} + ik\right)\cos(\hat{n}, R'')\right] da$$

If we now make the following feasible approximations in the optical case that:

1. $\frac{2\pi}{\lambda} >> \frac{1}{R'}$; $\frac{2\pi}{\lambda} >> \frac{1}{R''}$

2. cos. $(\hat{n}, R')$ and cos $(\hat{n}, R'')$ are constant over the aperture, i.e., cos $(\hat{n}, R') \cong 1$; cos $(\hat{n}, R'') \cong -1$

3. ϕ and $\frac{\partial \phi}{\partial n}$ vanish on the screen

we have:

2.11 $$\phi(x', y', z') = \frac{iA[\cos(\hat{n}, R') - \cos(\hat{n}, R'')]}{2\lambda R'R''}\iint_{apert.} e^{-ik(R'+R'')} da$$

and

2.12 $$\phi(x', y', z') = \frac{iA}{\lambda R'R''}\iint e^{-ik(R'+R'')} da$$

From Fig. 2-1, we have:

2.13 $$R' = \left[(x'+x)^2+(y'+y)^2+d'^2\right]^{1/2} = \left[\ell'^2+\rho'^2+2xx'+2yy'\right]^{1/2}$$

and $$R'' = \left[(x''-x)^2+(y''-y)^2+d''^2\right]^{1/2} = \left[\ell''^2+\rho''^2-2xx''-2yy''\right]^{1/2}$$

Expanding(2.13) in a power series we have:

2.14 $$R' = \ell'\left\{\left[1+\left(\frac{\rho'}{\ell'}\right)^2\right]^{1/2} + \frac{1}{\ell'^2}\left[1+\left(\frac{\rho'}{\ell'}\right)^2\right]^{-1/2}(xx'+yy')+\cdots\right\}$$

and $$R'' = \ell''\left\{\left[1+\left(\frac{\rho''}{\ell''}\right)^2\right]^{1/2} - \frac{1}{\ell''^2}\left[1+\left(\frac{\rho''}{\ell''}\right)^2\right]^{1/2}(xx''+yy'')+\cdots\right\}$$

Neglecting terms in $(\rho'/\ell')^2$ and $(\rho''/\ell'')^2$, and defining

$\mu' = \frac{x}{\ell'}$; $\mu'' = \frac{x}{\ell''}$ and $\nu' = \frac{y}{\ell'}$; $\nu'' = \frac{y}{\ell''}$

we have:

2.15 $$R'+R'' = \ell'+\ell''+(\mu'x'+\nu'y')-(\mu''x''+\nu''y'')$$

so that (2.12) becomes

2.16 $$\phi(x',y',z') = \frac{iAe^{-ik(\ell'+\ell'')}}{\lambda R'R''}\iint e^{-ik[(\mu'x'+\nu'y')-(\mu''x''+\nu''y'')]}\,da$$

We can simplify this expression further by defining reduced coordinates, i. e., let the maximum ang. coordinates be given by μ_0 and ν_0 in the x and y directions respectively; then let us define.

2.17 $$\beta = \frac{\mu'}{\mu_0} = \frac{x}{x_{max}} \; ; \quad \gamma = \frac{\nu'}{\nu_0} = \frac{y}{y_{max}}$$

and further let

2.18 $$\xi' = k\mu_0 x' \qquad \xi'' = k\mu_0 x''$$
$$\eta' = k\nu_0 y' \qquad \eta'' = k\nu_0 y''$$

then (2.16) becomes

2.19 $$\phi(\xi'\eta') = \frac{iAe^{-ik(l'+l'')}}{\lambda R'R''} \iint e^{i[\beta(\xi''-\xi')+\gamma(\eta''-\eta')]}\,da$$

We can further simplify this expression by normalizing the disturbance at $(\xi'=0=\eta')$ due to a point source at $(\xi''=0=\eta'')$, i. e. $\phi(0,0)=1$

And if we define:

$$X = \xi'' - \xi'$$
$$Y = \eta'' - \eta'$$

we have:

2.20 $$\phi(X,Y) = C\iint_{-\infty}^{\infty} e^{i(\beta X - \gamma Y)}\,d\beta\,d\gamma$$

For generality in the derivation we should postulate that due to aberrations, etc., a phase factor Δ should be introduced over the aperture so that writing this phase factor as

2.21 $$F(\beta,\gamma) = e^{i\frac{2\pi\beta}{\lambda}}$$ where $\Delta = \Delta(\beta,\gamma)$ is the deviation of the actual wave front from a reference sphere

(2.20) would become in general

2.22 $$\phi(X,Y) = \iint_{-\infty}^{\infty} F(\beta,\gamma)\, e^{i(\beta X+\gamma Y)}\,d\beta\,d\gamma$$ cf. fig 2-3

Thus we have the extremely important relationship that within the approximations made, the amplitude response at $P'(x',y')$ due to a point source at $P''(x'',y'')$ is, the Fourier transform of the phase function over the aperture.

For an ideal system, the phase factor is taken to be zero, i. e., $F(\beta,\gamma)=1$ over the aperture. Hence, in the absence of aberrations, the

amplitude response is the Fourier transform of the geometry of the aperture. This significant relationship is said to have originally been pointed out by Michelson. The notation and subsequent relationships in this paper, however, are to be found in the works of Duffieux and Maréchal.

Noting that the intensity distribution for a point source is given by:

2.23
$$I(X,Y) = \phi(X,Y)\,\phi^{*}(X,Y)$$

And recalling that the modulation function is defined as the Fourier transform of $I(X,Y)$, it can be shown that by making use of Equation (2.22) and 2.23, the modulation function can be obtained by performing a convolution integral of the aperture with itself, i. e., from

2.24
$$M(\beta',\gamma') = \iint_{-\infty}^{\infty} I(X,Y)\, e^{-i(\beta' X+\gamma' Y)}\, dX\, dY$$

it can be shown that*

2.25
$$M(\beta',\gamma') = \iint_{-\infty}^{\infty} F(\beta,\gamma)\, F^{*}(\beta+\beta',\gamma+\gamma')\, d\beta'\, d\gamma'$$

The importance of equation (2.25) in ideal cases will be demonstrated shortly, as the integral in equ. (2.24) is not always easily evaluated, and thus for apertures of simple geometry, equation (2.25) proves much more useful.

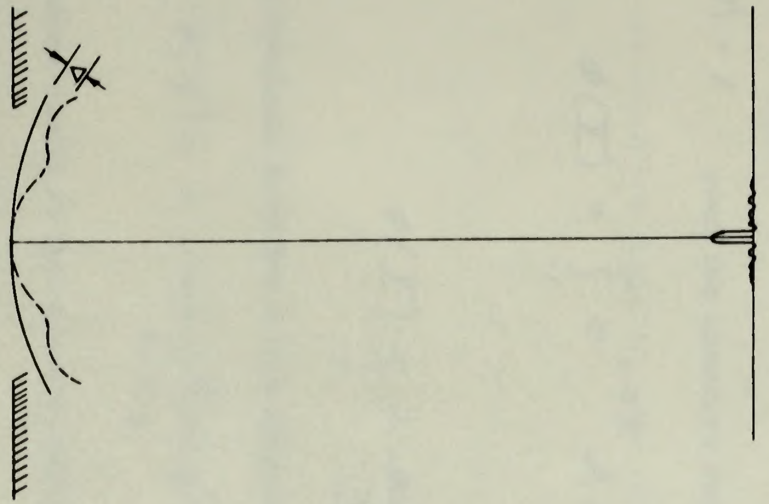
---

* From 2.24 and 2.23

$$M(\beta',\gamma') = \iint \left[\iiiint F(\beta,\gamma)\, F^{*}(\beta'',\gamma'')\, e^{i[(\beta-\beta'')X+(\gamma-\gamma'')Y]}\, d\beta\, d\gamma\, d\beta''\, d\gamma''\right] e^{-i(\beta' X+\gamma' Y)}\, dX\, dY$$

integrating first over $dX\ dY$ then over $d\beta''\ d\gamma''$

$$M(\beta',\gamma') = \iint_{-\infty}^{\infty} F(\beta,\gamma)\, F^{*}(\beta+\beta',\gamma+\gamma')\, d\beta,\, d\gamma$$



$$A(x,y) = \iint_{-\infty}^{\infty} F(\beta,\gamma)\, e^{i(\beta x + \gamma y)}\, d\beta\, d\gamma$$

$$I(x,y) = A(x,y)\; A^{*}(x,y)$$

$$M(\beta,\gamma) = \iint_{-\infty}^{\infty} I(x,y)\, e^{i(\beta x + \gamma y)}\, dx\, dy$$

$$M(\beta,\gamma) = \iint_{-\infty}^{\infty} F(\beta_0,\gamma_0)\, F^{*}(\beta+\beta_0, \gamma+\gamma_0)\, d\beta_0\, d\gamma_0$$

WHERE:

$$F(\beta,\gamma) = e^{ik\Delta}$$

$$\Delta = \Delta(\beta,\gamma)$$

Fig. 2-3

B. Single Slit Diffraction:

We shall now attempt to demonstrate the procedure outlined in the previous section for a one-dimensional problem, namely, the infinite one-dimensional slit. Equation 2.22 for the one-dimensional slit becomes

2.26
$$\phi(Y) = \int_{-\infty}^{\infty} F(\beta)\, e^{i\beta Y}\, d\beta$$

where we shall take

$F(\beta) = 1$ over the aperture and zero elsewhere; hence;

2.27
$$\phi(Y) = \int_{-1}^{1} e^{i\beta Y}\, d\beta = 2\int_{0}^{1} \cos \beta Y\, d\beta$$

or

2.28
$$\phi(Y) = 2\,\frac{\sin Y}{Y}$$

If we evaluate 2.28 for a line source at $y'' = 0$, we have,

2.29
$$\phi(y') = 2\,\frac{\sin(k V_0 y')}{k V_0 y'} = 2\,\frac{\sin\left(k \frac{d}{l} y'\right)}{\frac{k d y'}{l}}$$

It proves useful at this point to define $\omega_0 = \frac{kd}{l} = kV_0$, then in normalized form 2.29 becomes:

2.30
$$\phi(y') = \frac{\sin(\omega_0 y')}{\omega_0 y'}$$

and the intensity response to a line source is

2.31
$$I(y') = \left[\frac{\sin(\omega_0 y')}{\omega_0 y'}\right]^2$$

Let us first evaluate the modulation function by taking the Fourier transform of 2.31; later we shall show that the convolution integral method

due to Maréchal simplifies the procedure a great deal.

Thus

2.32 $$M(\omega) = \int_{-\infty}^{\infty} \left[\frac{\sin(\omega_0 y')}{\omega_0 y'}\right]^2 e^{i\omega y'}\, dy'$$

which is the Fourier transform of a triangular pulse about the origin. (A derivation of this is given in Appendix 2-I). In normalized form

2.33 $$M(\omega) = \left(1 - \frac{\omega}{2\omega_0}\right)$$ as shown in Figure 2-4

We see therefore that the upper limit of resolution of a sine target occurs for

2.34a $$\omega = 2\omega_0$$

or

2.34b $$\omega = \frac{2kd}{l} = \frac{2\pi}{P}$$

Hence a sine target whose period is less than $l\lambda / 2d$, or which subtends an angle

2.35 $$\theta < \frac{\lambda}{2d}$$

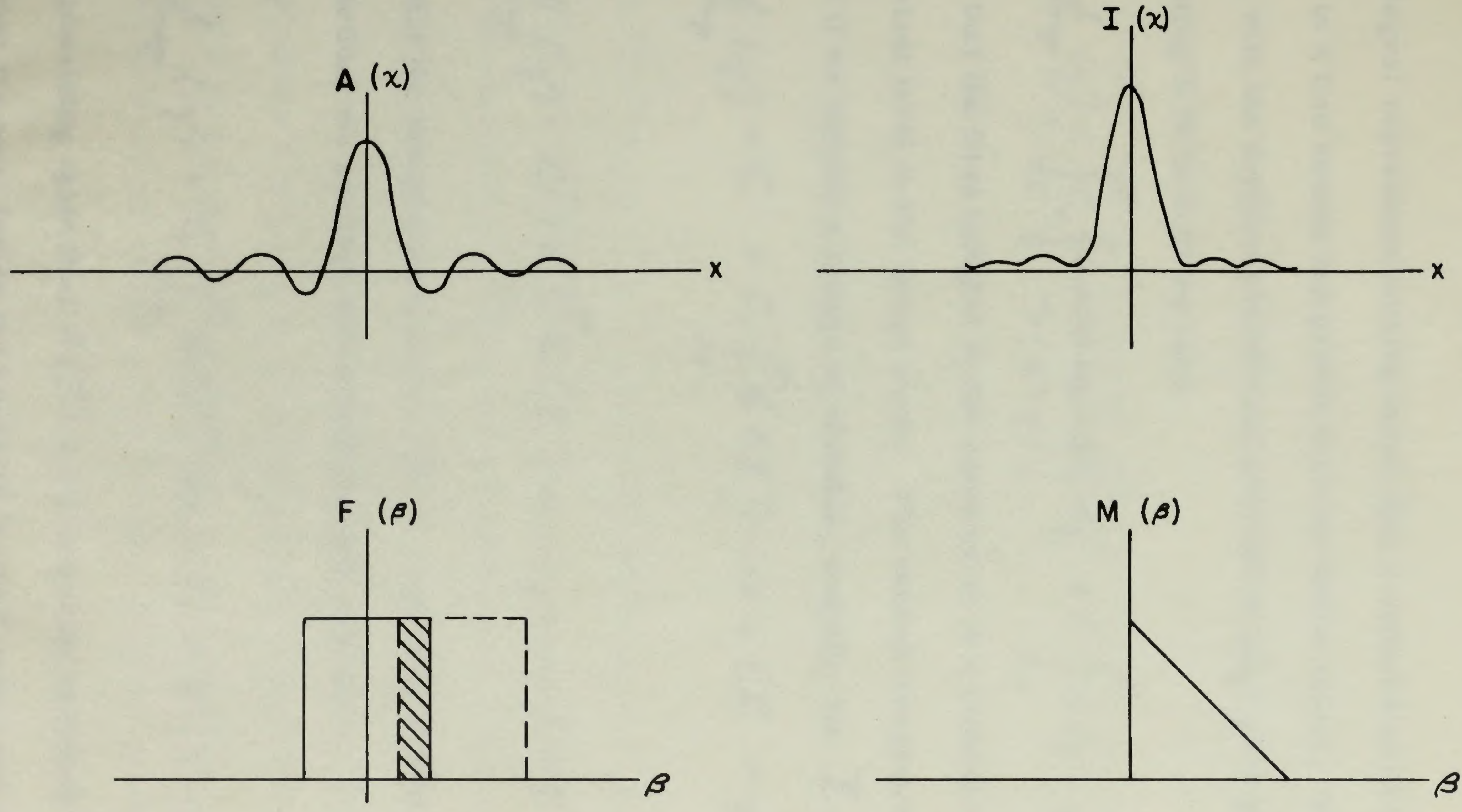
would not be resolved.

We can now give a simple interpretation of the modulation function as shown in fig. 2-4

Suppose that our object is a cosine target so that

2.36 $$f_{obj.}(y'') = C_0 + C_1 \cos \omega y''$$

then our image distribution is given by


A (x)
x
I (x)
x
F (β)
β
M (β)
β

SINGLE SLIT DIFFACTION

Fig. 2-4

2.37
$$f_{image}(y') = \int_{-\infty}^{\infty} h(y''-y')\, f_{obj.}(y'')\, dy''$$

This integral represents nothing more than a summation of the response of the system to a line source integrated over the whole object. We see that, in keeping with the notation established previously, $h(y''-y')$ is given by (equation 2. 31). Substituting 2. 36 in 2. 37 we have

2.38
$$f_{image}(y') = \int_{-\infty}^{\infty} C_0 \left[\frac{\sin \omega_0 (y''-y')}{\omega_0 (y''-y')}\right]^2 dy'' + C_1 \int_{-\infty}^{\infty} h(y''-y') \cos \omega y''\, dy''$$

We see that the first integral in the expression is a constant which establishes the constant level in the image plane. The second integral, however, is of interest if we employ a change of variable, namely, let $Y = y''-y'$ then:

2.39
$$f_{image}(y') = C_0' + C_1 \int_{-\infty}^{\infty} h(Y') \cos \omega (X' + y')\, dY'$$

or

2.40
$$f_{image}(y') = C_0' + C_1 \int_{-\infty}^{\infty} h(Y') \cos \omega y' \cos \omega Y\, dY' - C_1 \int_{-\infty}^{\infty} h(Y') \sin \omega y' \sin \omega Y'\, dY'$$

Noting that the integration is over Y', and that $h(Y')$ in our case is an even function, we see that the second integral vanishes.

Whence:

2.41
$$f_{image}(y') = C_0' \left[\int_{-\infty}^{\infty} h(Y') \cos \omega Y' \, dY'\right] C_1 \cos \omega y'$$

Emphasizing again that $h(Y')$ is the intensity response to a line source, we see that the term inside the brackets is it's Fourier transform. Since this is precisely the definition of the modulation function, equ. 2.41 becomes

2.42 $$f_{image}(y') = C_0' + M(\omega) C_1 \cos \omega y'$$

We can now interpret the modulation function in the following manner: given a cosine intensity distribution in the object, the amplitude will be reduced by a factor of $M(\omega)$ in the image plane, where $M(\omega)$ is the ordinate corresponding to a given ω in fig. 2-5. To generalize this still further, let us suppose that the object is a target whose Fourier expansion is of the form

2.43 $f_{obj.}(y'') = C_0 + C_1 \sin \omega y'' + C_3 \sin 3\omega y'' + C_5 \sin 5\omega y'' + \cdots$, i.e., a bar target.

We can write immediately that the image distribution is given by:

2.44 $$f_{image}(y') = C_0' + M(\omega) C_1 \sin \omega y + M(3\omega) C_3 \sin 3\omega y + \cdots$$

Now since $M(\omega) = 0$ for $\omega > 2\omega_0$, the diffraction system acts like a low-pass spatial frequency filter.

While the examples cited above were done by way of illustration, the importance of the method lies in the fact that the modulation function gives us directly a measure of amplitude reduction versus spatial frequency for any object expandable in a Fourier series or integral. Hence by properly defining contrast for a cosine target, and plotting the object spectrum together with the modulation function, proper multiplication will give the output (image) spectrum.

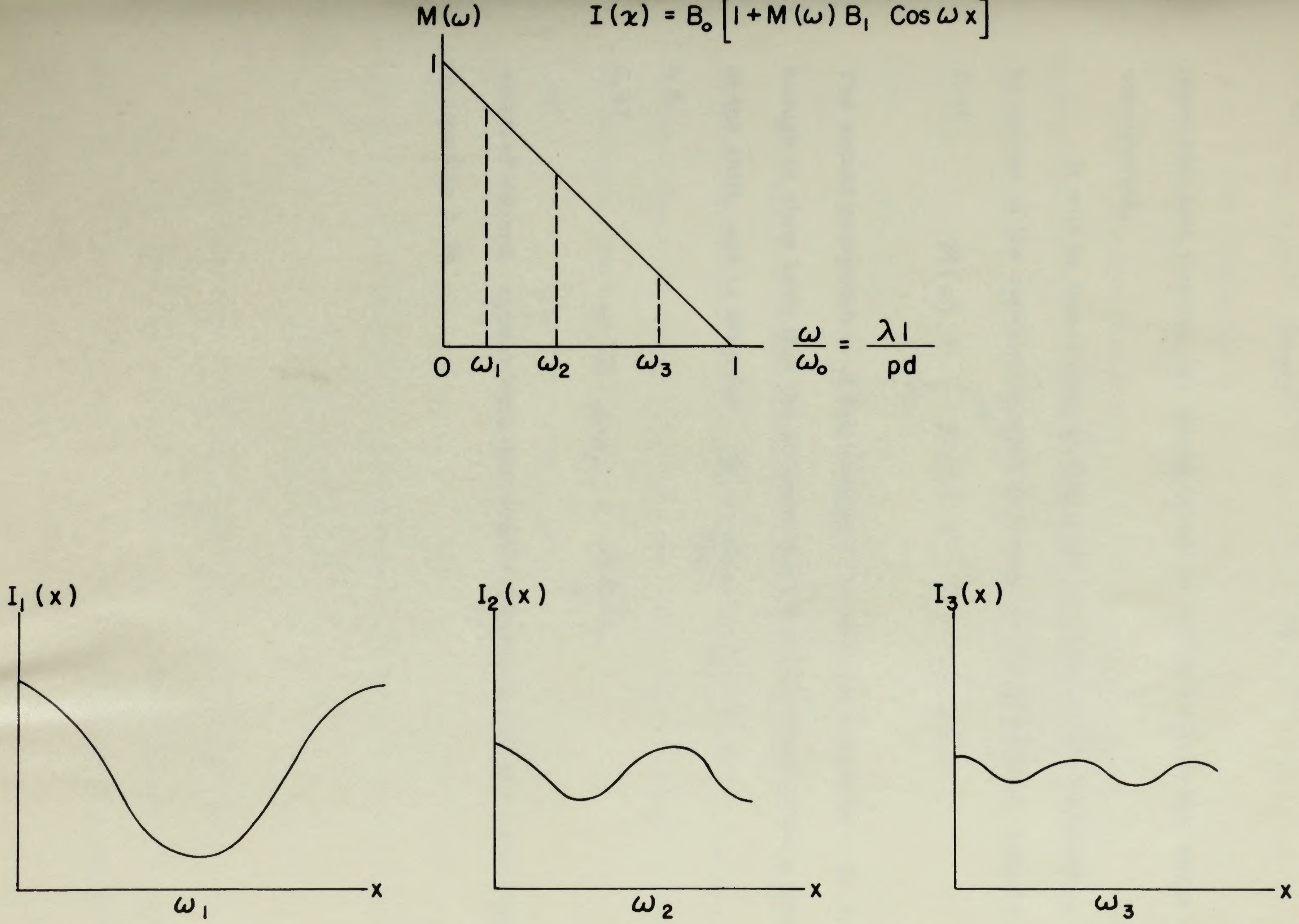

M(ω)
$I(x) = B_o\left[1 + M(\omega) B_1 \cos \omega x\right]$
1
0
ω_1
ω_2
ω_3
1
$\frac{\omega}{\omega_o} = \frac{\lambda l}{pd}$
$I_1(x)$
x
ω_1
$I_2(x)$
x
ω_2
$I_3(x)$
x
ω_3

Fig. 2-5

Since,

2.45 $$F_{image}(\omega) = M(\omega) F_{obj}(\omega)$$

from which, of course, the actual intensity distribution in the image can be constructed.

It will be interesting at this point to compare the modulation function by means of the convolution method suggested by Maréchal, namely:

2.46 $$M(\omega) = \int_{-\infty}^{\infty} F(\nu) F^{*}(\nu + \nu')\, d\nu$$

The actual computation of this integral is shown in Appendix 2-II; it is enough to state here that the common area falls off linearly as a function of the shift, and is zero for $\nu = \frac{y}{d/2} = \frac{\omega}{\omega_0} \geq 2$

i.e.

2.47 $$\omega \geq 2\omega_0 = \frac{2kd}{l}$$

which, of course, agrees with the results established earlier and given in equation 2.34.

C. Annulus and Circular Aperture:

The method for computing the modulation function for annuli and circular apertures is completely analogous to the previous treatment of the slit except that the problem is two-dimensional.

Let us imagine therefore a point source at the origin of our object plane, so far removed from the screen that essentially we can assume a plane wave incident upon the screen. With $x'' = 0 = y''$ our integral (equation 2.22) becomes

2.48
$$\phi(x', y') = \iint_{-\infty}^{\infty} F(\beta, \gamma)\, e^{-i(\beta\xi' + \gamma\eta')}\, d\beta\, d\gamma$$

Here again for a non-aberrating aperture, $F(\beta, \gamma) = 1$ over the annulus and zero else-where. We see further from the symmetry of the problem that (equation 2.48) can be written in terms of a Bessel transform as shown in (section I, equation 1.80), i. e.

2.49
$$\phi(\rho') = 2\pi \int_{\sigma_1}^{\sigma_2} J_0(\rho'\sigma)\, \sigma\, d\sigma$$

where $\sigma^2 = \beta^2 + \gamma^2$ $\quad \rho'^2 = \xi'^2 + \eta'^2$

making use of the recurrence relationship for Bessel functions we find that:

$$\frac{d}{dx}\left(x^n J_n\right) = x^n J_{n-1}$$

2.50
$$\phi(\rho') = 2\pi\sigma_2^2 \left[\frac{J_1(\rho'\sigma_2)}{\rho'\sigma_2} - \alpha^2 \frac{J_1(\rho'\sigma_1)}{\rho'\sigma_1}\right]$$

and $\alpha = \frac{\sigma_1}{\sigma_2} = \frac{R_1}{R_2}$ where R_1 and R_2 are the inner and outer radii respectively

hence $\sigma = \sqrt{\frac{x^2 + y^2}{R_2^2}} = \frac{r}{R_2}$

2.51 $$\sigma_1 = \frac{R_1}{R_2} = \alpha \; ; \qquad \sigma_2 = \frac{R_2}{R_2} = 1$$

and 2.50 becomes:

2.52 $$\phi(\rho') = 2\pi\left[\frac{J_1(\rho')}{\rho'} - \alpha^2\,\frac{J_1(\alpha\rho')}{\alpha\rho'}\right]$$

1) Circular Aperture

At this point, for the sake of clarity, let us consider the case of a circular aperture of radius R, $(\alpha = 0)$ we then have in normalized form

2.53 $$\phi(\rho') = \frac{2J_1\left(\frac{kR}{l}p'\right)}{\left(\frac{kR}{l}p'\right)}$$

where $$\rho' = \frac{kR}{l}\sqrt{x'^2 + y'^2} = \frac{2\pi R}{\lambda l}p'$$

Further, let us define

2.54 $$\omega_0 = \frac{kR}{l} = \frac{2\pi R}{\lambda l}$$

The reason for this choice will become evident later on in this discussion. Equation 2.53 is the normalized amplitude response of the system to a unit impulse (point source). The intensity response therefore is given by

2.55 $$h(p') = \left[\frac{2J_1(\omega_0 p')}{\omega_0 p'}\right]^2$$

where $p'^2 = x'^2 + y'^2$ in the image plane. The modulation function $M(\omega)$ can be evaluated one of two ways; either by a direct Fourier Transform of equation 2.55, or by the convolution integral over the aperture as suggested by Marechal*. In this case it proves simpler to perform the convolution

---

* cf. (ref. 6)

over the aperture as shown in Appendix 2-III, the result being

2.56 $$M(\omega) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{\omega}{2\omega_0}\right) - \frac{\omega}{2\omega_0}\sqrt{1-\left(\frac{\omega}{2\omega_0}\right)^2}\,\right]$$

While this result was arrived at rather simply by geometry, it is instructive to compute the modulation function analytically. This has been done by Steel (ref. 13), and will be outlined here in order to illustrate the method by which one can compute modulation functions.

The amplitude response at $P'(x', y')$ due to a point source at $P''(x'', y'')$ is given in terms of equation 2.22, where, upon integration in precisely the same manner as in equation 2.49, we have

2.57 $$\phi(X, Y) = \frac{J_1\left(\sqrt{X^2+Y^2}\right)}{\sqrt{X^2+Y^2}}$$

where again

2.58 $$X = \xi'' - \xi' = k u_0 (x''-x') = \omega_{x_0}(x''-x')$$
$$Y = \eta'' - \eta' = k v_0 (y''-y') = \omega_{y_0}(y''-y')$$

Hence the intensity response at $x' = 0$, due to a line source along $y'' = 0$, is given by

2.59 $$L(\eta') = \int_{-\infty}^{\infty} \frac{J_1\sqrt{\xi''^2+\eta'^2}}{\sqrt{\xi''^2+\eta'^2}}\, d\xi''$$

employing a change of variable; $t^2 = \xi''^2 + \eta'^2$

2.60 $$L(\eta') = 2\int_{\eta'}^{\infty} \frac{J_1(t)}{t^2\sqrt{t^2-\eta'^2}}\, dt$$

the result of which is

2.61 $$L(\eta') = \frac{H_1(2\eta')}{\eta'^2}$$

where $H_1(2\eta')$ is the Struve function of first order and is defined as

2.62 $$H_1(2\eta') = \frac{4\eta'}{\pi}\int_0^{\pi} \sin(2\eta'\sin\varphi)\cos^2\varphi \, d\varphi$$

Hence in normalized form the intensity response of the system to a line source is

2.63 $$L(\eta') = \frac{3\pi}{8}\,\frac{H_1(2\eta')}{\eta'^2}$$

The modulation function is then the Fourier transform of 2.63 as shown by Steel (ref. 13); namely

2.64 $$M(\omega) = \frac{3\pi}{8}\int_{-\infty}^{\infty} \frac{H_1(2\omega_0 y')}{(\omega_0 y')^2}\, e^{i\omega y'}\, dy'$$

or in terms of $\eta' = \omega_0 y'$, we have

2.65 $$M(\omega) = \frac{3\pi}{4\omega_0}\int_0^{\infty} \frac{H_1(2\eta')}{\eta'^2}\cos\left(\frac{\omega\eta'}{\omega_0}\right) d\eta'$$

Neglecting the constant for the time being, and making use of the definition of the Struve function, we have

2.66 $$\int_0^{\infty}\frac{H_1(2\eta')}{\eta'^2}\cos\left(\frac{\omega\eta'}{\omega_0}\right) d\eta' = \frac{4}{\pi}\int_0^{\pi/2}\cos^2\varphi\, d\varphi \int_0^{\infty}\frac{\cos\frac{\omega\eta'}{\omega_0}\,\sin(2\eta'\sin\varphi)}{\eta}\, d\eta$$

Making use of the relationship between the sine of the sum and difference of two angles;

2.67 $$M(\omega) = \frac{3}{2\omega_0}\int_0^{\pi/2}\cos^2\varphi\, d\varphi \int_0^{\infty}\frac{\sin\left[2\left(\frac{\omega}{2\omega_0}+\sin\varphi\right)\eta\right]}{\eta} - \frac{\sin\left[2\left(\frac{\omega}{2\omega_0}-\sin\varphi\right)\eta\right]}{\eta}\, d\eta$$

but

2.68 $$\int_0^{\infty}\frac{\sin ax}{x}\, dx = \begin{cases} \frac{\pi}{2} & a > 0 \\ 0 & a = 0 \\ -\frac{\pi}{2} & a < 0 \end{cases}$$

From this integral we see that

2.69 $$\text{if} \quad \frac{\omega}{2\omega_0} \geq 1 \qquad \text{then} \quad \frac{\omega}{2\omega_0} + \sin\theta > 0$$

$$\text{and} \quad \frac{\omega}{2\omega_0} - \sin\theta > 0$$

$$\text{if} \quad \frac{\omega}{2\omega_0} < 1 \qquad \left(\frac{\omega}{2\omega_0} + \sin\theta\right) \text{ and } \left(\frac{\omega}{2\omega_0} - \sin\theta\right)$$

will be of opposite signs in the interval; $\sin^{-1}\frac{\omega}{2\omega_0} < \varphi < \frac{\pi}{2}$

hence

2.70 $$M(\omega) = \frac{3\pi}{2\omega_0}\int_{\sin^{-1}\frac{\omega}{2\omega_0}}^{\pi/2} \cos^2\varphi \, d\varphi \qquad \frac{\omega}{2\omega_0} < 1$$

$$= 0 \qquad \frac{\omega}{2\omega_0} > 1$$

Hence our normalized, one-dimensional modulation function becomes

2.71 $$M(\omega) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{\omega}{2\omega_0}\right) - \frac{\omega}{2\omega_0}\sqrt{1 - \left(\frac{\omega}{2\omega_0}\right)^2}\right] \qquad \frac{\omega}{2\omega_0} \leq 1$$

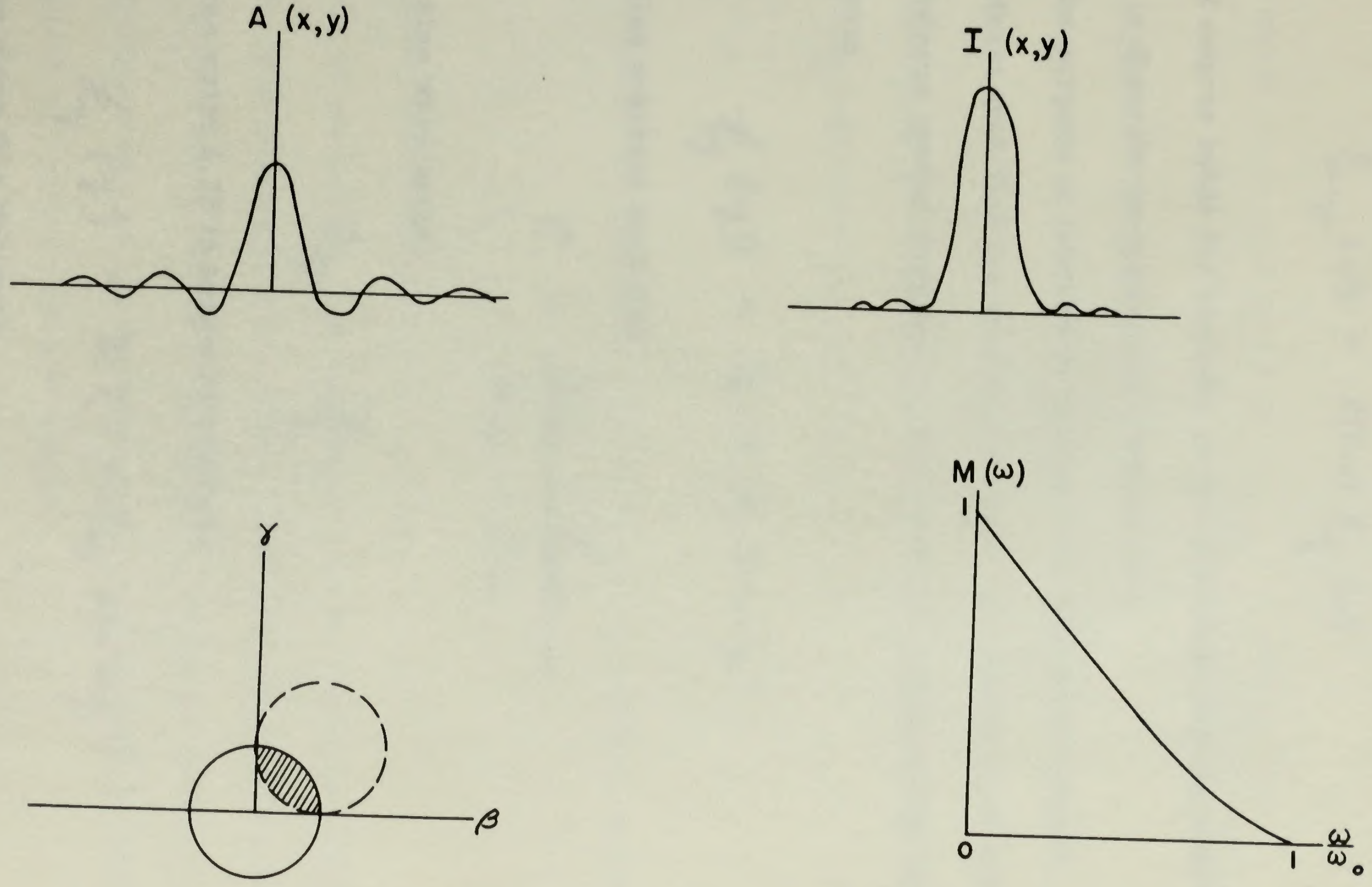
which of course agrees with equation 2.56.

The interpretation of the modulation function will again be emphasized at this point. From the curve shown in Fig. 2-6, we see that a sine wave object whose spatial frequency is such that

2.72 $$\omega > 2\omega_0 = \frac{2kR}{l}$$ will not be passed by the system

This corresponds to a limit of resolution for the sine wave in terms of the period

2.73 $$\theta = \frac{P}{l} = \frac{\lambda}{2R}$$


A (x,y)
I (x,y)
γ
β
M (ω)
1
0
1
ω/ω₀

DIFFRACTION BY A CIRCULAR APERTURE

Fig. 2-6

Thus the spatial spectrum can be evaluated from our relationship previously established.

2.74
$$F_{image}(\omega) = M(\omega)\, F_{obj.}(\omega)$$

and this of course holds for periodic or for transient objects whether their spectrum is discrete or continuous, respectively.

For the purpose of laboratory testing with sine wave targets, one can define contrast such that the modulation curve is a measure of contrast rendition versus spatial frequency. For example, suppose that our target is of the form

2.75
$$f_{obj.}(y'') = B_0 + B_1 \cos \omega y''$$

and we define contrast such that

2.76
$$C = \frac{I_{max.} - I_{min.}}{I_{max.} + I_{min.}}$$

then for a sine wave target

2.77
$$C_{obj.} = \frac{B_1}{B_0}$$

hence we can write 2.75 in terms of contrast

2.78
$$f_{obj.}(y'') = B_0 \left(1 + C_{obj.} \cos \omega y''\right)$$

In the image plane this becomes

2.79
$$f_{image}(y') = B_0' \left(1 + M(\omega) \frac{B_1}{B_0} \cos \omega y'\right)$$

or

2.80
$$f_{image}(y') = B_o'(1 + C_{image}\cos\omega y')$$

where the image contrast is given by $C_{image} = M(\omega)\dfrac{B_1}{B_o'}$

If we now define the contrast rendition as

2.81
$$C_R = \frac{C_{image}}{C_{obj}}.$$

we can indeed speak of the modulation curve as a contrast rendition versus spatial frequency plot for sine wave targets; for

2.82
$$C_R = M(\omega)\frac{B_o}{B_o'}$$

and since B_o and B_o' are related by a proportionality constant, we have that

2.83
$$C_R \sim M(\omega)$$

which of course can be normalized to one for $\omega = 0$.

It should be pointed out, however, that now for sinusoidal objects in general, the modulation curve is a direct measure of the amplitude reduction of the sine and/or cosine terms in the Fourier series of the object in the sense of equation 2.44. In the terminology of network theory, diffracting systems behave as low-pass spatial filters.

It is to be noted that in the Maréchal scheme and Steel's method for evaluating the modulation function, the integrations and results were basically one-dimensional. For example, Steel found the response of the system to a line source and took the one-dimensional Fourier transform.

For the point source response, we shall take the double Fourier transform with circular symmetry. The result will be a two-dimensional modulation function which of course should reduce to the one-dimensional for a sine wave with a frequency component in one direction.

The method for performing the integration was suggested to me by Dr. Hauser. The double Fourier transform of equation 2.55 is,

2.84 $$M(\omega_x,\omega_y) = \iint_{-\infty}^{\infty} \left[\frac{J_1\left(\omega_0\sqrt{x'^2+y'^2}\right)}{\omega_0\sqrt{x'^2+y'^2}}\right]^2 e^{-i(\omega_x x'+\omega_y y')}\, dx'\, dy'$$

transforming to polar coordinates

2.85 $$M(\omega_r) = \frac{1}{\omega_0^2}\int_0^{\infty}\int_0^{2\pi} \frac{J_1^2(\rho')}{\rho'^2}\, e^{-i\sigma\rho'\cos\theta}\, \rho'\, d\rho'\, d\theta$$

where of course

$$\sigma = \frac{\omega_r}{\omega_0} \qquad \rho' = \omega_0 r'$$

Let us now concentrate on the integral

2.86 $$I = \int_0^{\infty}\int_0^{2\pi} \frac{J_1^2(\rho')}{\rho'^2}\, e^{-i\sigma\rho'\cos\theta}\, \rho'\, d\rho'\, d\theta$$

or in the form

2.87 $$I = \int_0^{\infty}\int_0^{2\pi} J_1^2(\rho')\left[\frac{e^{-i\sigma\rho'\cos\theta}}{-i\rho'\cos\theta}\right](-i\cos\theta)\, d\rho'\, d\theta$$

The term inside the brackets can be expressed as an integral over σ since:

2.88 $$\int_0^{u} e^{-au}\, du = \frac{e^{-au}}{-a} + \frac{1}{a}$$

Thus we can write

2.89 $$I = -i\int_0^{\infty}\int_0^{2\pi}\int_0^{\sigma} J_1^2(\rho')\, e^{-i\sigma\rho'\cos\theta} \cos\theta\, d\rho'\, d\theta\, d\sigma + \int_0^{\infty}\int_0^{2\pi} \frac{J_1^2(\rho')}{\rho'}\, d\rho'\, d\theta$$

Let

2.90 $$I_1 = -i\int_0^\infty d\rho' \int_0^\alpha d\sigma \int_0^{2\pi} J_1^2(\rho')\, e^{-i\sigma\rho'\cos\theta} \cos\theta\, d\theta$$

and

2.91 $$I_2 = \int_0^\infty \int_0^{2\pi} \frac{J_1^2(\rho')}{\rho'}\, d\rho'\, d\theta$$

The integral in equation 2.91 is given in Watson's Bessel Functions , p. 405, as

2.92 $$I_2 = 2\pi \int_0^\infty \frac{J_1^2(\rho')}{\rho'}\, d\rho' = \pi$$

As for I_1, making use of formula #5, p. 20, Watson's Bessel Functions , for the integral def. of the Bessel Function of order n, that

2.93 $$J_n(z) = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} e^{i(n\theta - z\sin\theta)}\, d\theta = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} e^{-it\cos\theta} \cos n\theta\, d\theta$$

which we have for the inner integral over θ in equation 2.90

2.94 $$\int_0^{2\pi} e^{-i\sigma\rho'\cos\theta} \cos\theta\, d\theta = -2\pi i\, J_1(\sigma\rho')$$

hence

2.95 $$I_1 = -2\pi \int_0^\sigma d\sigma \int_0^\infty J_1(\rho')\, J_1(\rho')\, J_1(\sigma\rho')\, d\rho'$$

Again referring to Watson's Bessel Functions, #3, P. 411, we find that

2.96 $$\int_0^\infty J_\nu(at)\, J_\nu(bt)\, \frac{J_\nu(ct)}{t^{\nu-1}}\, dt = \frac{2^{\nu-1}\, \Delta^{2\nu-1}}{(abc)^\nu\, \Gamma(\nu+\frac{1}{2})\, \Gamma(\frac{1}{2})}$$

if a, b, c are the sides of a triangle whose area is Δ ; but that if a, b, c are not the sides of a triangle then the integral is zero. In our case Δ is

an isosceles triangle with unit sides and base σ , and hence the integral will be zero for $\sigma > 2$. Thus for the integral over ρ' in 2.95 we have

2.97
$$\int_0^\infty J_1(\rho')J_1(\rho')J_1(\alpha\rho')\,d\rho' = \frac{\Delta}{\sigma\,\Gamma(3/2)\,\Gamma(1/2)} \qquad 0 \le \sigma \le 2$$

where the Γ functions are given in general by

2.98
$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx \qquad n > 0$$

Integration by parts gives:

2.99
$$\Gamma(1+n) = n\,\Gamma(n)$$

if we let $x = y^2$ in 2.98 we have

2.100
$$\Gamma(n) = 2\int_0^\infty y^{2n-1} e^{-y^2}\,dy$$

and for $n = 1/2$

2.101
$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-y^2}\,dy = \sqrt{\pi}$$

and by 2.99

2.102
$$\Gamma(\tfrac{3}{2}) = \Gamma(1+\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}$$

We have also for Δ , the area of the isosceles triangle of base σ , and having unit sides:

2.103
$$\Delta = \frac{1}{2}\sigma\sqrt{1-\left(\frac{\sigma}{2}\right)^2}$$

Thus 2.95 becomes

2.104 $$I_1 = -2\pi \int_0^{\sigma} \frac{2}{\pi\sigma}\left(\frac{\sigma}{2}\sqrt{1-\left(\frac{\sigma}{2}\right)^2}\right. d\sigma \qquad 0 \le \sigma \le 2$$

or in neater form

2.105* $$I_1 = -2\int_0^{\sigma} \sqrt{1-\left(\frac{\sigma}{2}\right)^2}\, d\sigma \qquad 0 \le \sigma \le 2$$

Now let $\cos x = \frac{\sigma}{2}$; $\qquad -\sin x\, dx = \frac{d\sigma}{2}$

$$\sqrt{1-\left(\frac{\sigma}{2}\right)^2} = \sin x$$

and our integral becomes

2.106 $$I_1 = 4\int_{\pi/2}^{\cos^{-1}\frac{\sigma}{2}} \sin^2 x\, dx = -4\int_{\cos^{-1}\frac{\sigma}{2}}^{\pi/2} \sin^2 x\, dx$$

---

* A note is in order here with regard to the value of I_2 for $\sigma > 2$

Note that for $\sigma > 2$ we can write equation 2.105 in the form

$$I_1 = -4\int_0^{\sigma} \frac{\Delta}{\sigma}\, d\sigma$$

$$= -4\int_0^{2} \frac{\Delta}{\sigma}\, d\sigma - 4\int_2 \frac{\Delta}{\sigma}\, d\sigma \qquad \sigma > 2$$

However the second integral is zero since $\Delta = 0$ for $\sigma > 2$,

hence $$I_1 = -2\int_0^{2} \sqrt{1-\left(\frac{\sigma}{2}\right)^2}\, d\sigma \qquad \sigma > 2$$

$$= -2\left[\frac{\sigma}{2}\sqrt{1-\left(\frac{\sigma}{2}\right)^2} + \sin^{-1}\left(\frac{\sigma}{2}\right)\right]_0^2 \qquad \sigma > 2$$

$$= -\pi$$

hence

$$I = I_1 + I_2 = -\pi + \pi = 0 \qquad \sigma > 2$$

hence $$I = 0 \qquad \sigma > 2$$

or

2.107 $$I_1 = 2\left[\cos^{-1}\frac{\sigma}{2} - \frac{\sigma}{2}\sqrt{1-\left(\frac{\sigma}{2}\right)^2} - \frac{\pi}{2}\right] \qquad 0 \le \sigma \le 2$$

Hence equation 2.89, which is the sum of I_1 and I_2, becomes

2.108 $$\begin{aligned} I &= 2\left[\cos^{-1}\frac{\sigma}{2} - \frac{\sigma}{2}\sqrt{1-\left(\frac{\sigma}{2}\right)^2}\right] && 0 \le \sigma \le 2 \\ &= 0 && \sigma > 2 \end{aligned}$$

and recalling that $\sigma = {}^{\omega_r}/_{\omega_{0r}}$ our normalized two-dimensional modulation function is

2.109 $$M(\omega_r) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{\omega_r}{2\omega_{0r}}\right) - \frac{\omega_r}{2\omega_{0r}}\sqrt{1-\left(\frac{\omega_r}{2\omega_{0r}}\right)^2}\right] \qquad \frac{\omega}{\omega_0} \le 2$$

We see of course that the two-dimensional modulation function with circular symmetry is just the one-dimensional function generated through 2π, and that when our object is one-dimensional with a frequency component, say along ω_x only, then $\omega_r = \omega_x$, and we have the equivalent one-dimensional modulation function for a circular aperture.

2) Annuli

Recalling that the amplitude response to a point source is given in equation 2.52, we have for the normalized intensity response

2.110
$$h(\rho') = \left[\frac{J_1(\rho')}{\rho'}\right]^2 + \alpha^4\left[\frac{J_1(\alpha\rho')}{\alpha\rho'}\right]^2 - 2\alpha\left[\frac{J_1(\rho')\,J_1(\alpha\rho')}{\rho'^2}\right] \; ; \quad \alpha = \frac{R_1}{R_2}$$

Taking the two-dimensional transform in polar form we have for the modulation function

2.111
$$M(\omega_r) = \int_0^{\infty}\int_0^{2\pi}\left[\frac{J_1(\rho')}{\rho'}\right]^2 e^{-i\sigma\rho'\cos\theta\, d\theta}\,\rho'\,d\rho'\,d\theta$$
$$+\,\alpha^2\int_0^{\infty}\int_0^{2\pi}\left[\frac{J_1(\alpha\rho')}{\alpha\rho'}\right]^2 e^{-i\frac{\sigma}{\alpha}(\alpha\rho')\cos\theta}\,(\alpha\rho')\,d(\alpha\rho')\,d\theta$$
$$-\,2\alpha\int_0^{\infty}\int_0^{2\pi}\frac{J_1(\rho')\,J_1(\alpha\rho')}{\rho'^2}\,e^{-i\sigma\rho'\cos\theta}\,\rho'\,d\rho'\,d\theta$$

The first two integrals in equation 2.111 (which we shall designate as I_1 and I_2 respectively) have already been evaluated for the circular aperture, the solutions of which are

2.112*
$$I_1 = \frac{2}{\pi}\left[\cos^{-1}\frac{\omega}{2\omega_0} - \frac{\omega}{2\omega_0}\sqrt{1-\left(\frac{\omega}{2\omega_0}\right)^2}\,\right] \qquad 0 \le \frac{\omega}{2\omega_0} \le 1$$

and

2.113
$$I_2 = \frac{2\alpha^2}{\pi}\left[\cos^{-1}\frac{\omega}{2\alpha\omega_0} - \frac{\omega}{2\alpha\omega_0}\sqrt{1-\left(\frac{\omega}{2\alpha\omega_0}\right)^2}\,\right] \qquad 0 \le \frac{\omega}{2\omega_0} \le \alpha$$

we thus have the integral

2.114
$$I_3 = -\frac{2\alpha}{\pi}\int_0^{\infty}\int_0^{2\pi}\frac{J_1(\rho')\,J_1(\alpha\rho')}{\rho'^2}\,e^{-i\rho'\sigma\cos\theta}\,\rho'\,d\rho'\,d\theta$$

---

*all terms have been multiplied by ω_0^2/π for normalization purposes

yet to evaluate

Again making use of the scheme of equations 2.87, 2.88 and 2.89, i. e., noting that

2.115
$$\frac{e^{-i\rho'\sigma\cos\theta}}{-i\rho'\cos\theta} = \int_0^\sigma e^{-i\rho'\cos\theta\,\sigma}\,d\sigma - \frac{1}{i\rho'\cos\theta}$$

we have for 2.114

2.116
$$I_3 = \frac{2\alpha i}{\pi}\int_0^\infty d\rho'\int_0^\sigma d\sigma\int_0^{2\pi} J_1(\rho')J_1(\alpha\rho')e^{-i\sigma\rho'\cos\theta}\cos\theta\,d\theta - \frac{2\alpha}{\pi}\int_0^\infty d\rho'\int_0^{2\pi}\frac{J_1(\rho')J_1(\alpha\rho')}{\rho'}\,d\theta$$

Now defining:

2.117
$$I_4 = \frac{2\alpha i}{\pi}\int_0^\infty d\rho'\int_0^\sigma d\sigma\int_0^{2\pi} J_1(\rho')J_1(\alpha\rho')\,e^{-i\sigma\rho'\cos\theta}\cos\theta\,d\theta$$

and

2.118
$$I_5 = -\frac{2\alpha}{\pi}\int_0^\infty d\rho'\int_0^{2\pi}\frac{J_1(\rho')J_1(\alpha\rho')}{\rho'}\,d\theta$$

Performing the integrations over θ and making use of equation 2.94

2.119
$$I_4 = 4\alpha\int_0^\sigma d\sigma\int_0^\infty J_1(\rho')\,J_1(\alpha\rho')\,J_1(\sigma\rho')\,d\rho'$$

and

2.120
$$I_5 = -4\alpha\int_0^\infty \frac{J_1(\rho')\,J_1(\alpha\rho')}{\rho'}\,d\rho'$$

Again referring to Watson's Bessel Functions, p. 405, we have that:

2.121
$$\int_0^\infty \frac{J_1(\rho')\,J_1(\alpha\rho')}{\rho'}\,d\rho' = \begin{cases}\frac{\alpha}{2} & \alpha \leq 1\\ \frac{1}{2\alpha} & \alpha \geq 1\end{cases}$$

Since, from the physical interpretation of α in this problem, it can only

take on values between 0 and 1, we shall accept only the first of these values.

Hence:

2.122 $$I_5 = -2\alpha^2$$

Referring back to equation 2.96 and its interpretation, we have

2.123 $$I_4 = 4\alpha \int_0^{\sigma} \frac{\Delta}{2\sigma \, \pi/2} \, d\sigma$$

where, here, Δ is the area of a triangle of base σ and sides α and 1. In terms of the angle included between the sides 1 and α we have that

2.124 $$\Delta = \frac{1}{2} \alpha \sin \phi$$

$$\sigma^2 = 1 + \alpha^2 - 2\alpha \cos \phi$$

$$\sigma \, d\sigma = \alpha \sin \phi \, d\phi$$

and again we note that Δ is zero everywhere except between the limits $1-\alpha \leq \sigma \leq 1+\alpha$ so that

2.125 $$I_4 = \frac{4\alpha^2}{\pi} \int_0^{\phi_0} \frac{\sin^2 \phi \, d\phi}{1 + \alpha^2 - 2\alpha \cos \phi}$$

where $$\phi_0 = \cos^{-1} \frac{1 + \alpha^2 - \sigma^2}{2\alpha}$$

It proves convenient to write equation 2.125 in the form

2.126 $$I_4 = \frac{4\alpha^2}{\pi b} \int_0^{\phi_0} \frac{\sin^2 \phi \, d\phi}{c + \cos \phi} \quad \text{where} \quad b = -2\alpha, \quad c = \frac{1+\alpha^2}{-2\alpha}$$

and

2.127 $$I_4 = \frac{4\alpha^2}{\pi b} \int_0^{\phi_0} \frac{(1-c^2)\, d\phi}{c + \cos \phi} + \frac{4\alpha^2}{\pi b} \int_0^{\phi_0} \frac{(c^2 - \cos^2 \phi)}{c + \cos \phi} \, d\phi$$

whence, upon integration:

2.128 $$I_4 = \frac{4\alpha^2}{\pi b}\left[\frac{2(1-c^2)}{\sqrt{c^2-1}}\tan^{-1}\frac{\sqrt{c^2-1}\,\tan \phi_0/2}{c+1} + c\,\phi_0 - \sin\phi_0\right]$$

or in terms of α

2.129 $$I_4 = \frac{2}{\pi}\left[(\alpha^2-1)\tan^{-1}\left(\frac{1+\alpha}{1-\alpha}\tan^{-1}\frac{\phi_0}{2}\right) + \left(\frac{1+\alpha^2}{2}\phi_0 + \alpha\sin\phi_0\right)\right]$$

Now combining equation 2.129 with 2.122, we have

2.130 $$I_3 = -2\alpha^2 + \frac{2}{\pi}\left[\left(\frac{1+\alpha^2}{2}\right)\phi_0 + \alpha\sin\phi_0 - (1-\alpha^2)\tan^{-1}\left(\frac{1+\alpha}{1-\alpha}\tan\frac{\phi_0}{2}\right)\right]$$

where the term inside the brackets is defined only for $1-\alpha \leq \sigma \leq 1+\alpha$

For $\sigma > 1+\alpha$ we can write 2.123 as

2.131 $$I_4 = \frac{8}{\pi}\int_0^{1+\alpha}\frac{\Delta}{\sigma}\,d\sigma + \frac{8}{\pi}\int_{1+\alpha}^{\sigma}\frac{\Delta}{\sigma}\,d\sigma$$

where we see that the second integral is zero, since Δ is zero. Writing the first integral in terms of ϕ, where $\phi_0 = \pi$, for $\sigma = 1+\alpha$, we have from 2.129 with $\phi_0 = \pi$.

2.132 $$I_4 = 2\alpha^2 \qquad\qquad \sigma > 1+\alpha$$

whence:

2.133 $$I_3 = I_4 + I_5 = 0 \qquad\qquad \sigma > 1+\alpha$$

Thus in terms of I_1, I_2, and I_3, we have for the modulation function of the annulus from equation 2.111.

2.134 $$M(\omega_r) = I_1 + I_2 - I_3$$

where

$$I_1 = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{\sigma}{2}\right) - \left(\frac{\sigma}{2}\right)\sqrt{1-\left(\frac{\sigma}{2}\right)^2}\right] \qquad 0 \le \frac{\sigma}{2} \le 1$$

$$I_2 = \frac{2\alpha^2}{\pi}\left[\cos^{-1}\left(\frac{\sigma}{2\alpha}\right) - \frac{\sigma}{2\alpha}\sqrt{1-\left(\frac{\sigma}{2\alpha}\right)^2}\right] \qquad 0 \le \frac{\sigma}{2} \le \alpha$$

$$I_3 = \begin{cases} 2\alpha^2 & 0 \le \frac{\sigma}{2} \le \frac{1-\alpha}{2} \\ 2\alpha^2 - \left(\frac{1+\alpha^2}{\pi}\right)\phi_0 - \frac{2\alpha}{\pi}\sin\phi_0 + \frac{2(1-\alpha^2)}{\pi}\tan^{-1}\left(\frac{1+\alpha}{1-\alpha}\tan\frac{\phi_0}{2}\right) & \frac{1-\alpha}{2} \le \frac{\sigma}{2} \le \frac{1+\alpha}{2} \\ 0 & \frac{\sigma}{2} \ge \frac{1+\alpha}{2} \end{cases}$$

and

$$\phi_0 = \cos^{-1}\frac{1+\alpha^2-\sigma^2}{2\alpha} \; ; \quad \sigma = \frac{\omega_x}{\omega_{0x}} \; ; \quad \alpha = \frac{R_1}{R_2}$$

as defined in equation 2.50

Again it should be pointed out that if one is dealing with objects which have only one frequency component, so that $\omega_x = \omega_y$ for example, then equation 2.134 reduces to Steel's result in ref. 13, as indeed it should. It is possible also to evaluate the modulation function by means of a convolution integral. The actual scheme for doing this geometrically is shown in Appendix 2-IV and the modulation function for $\alpha = 0, 1/4, 1/2, 3/4$ is given in fig. 2-7. It

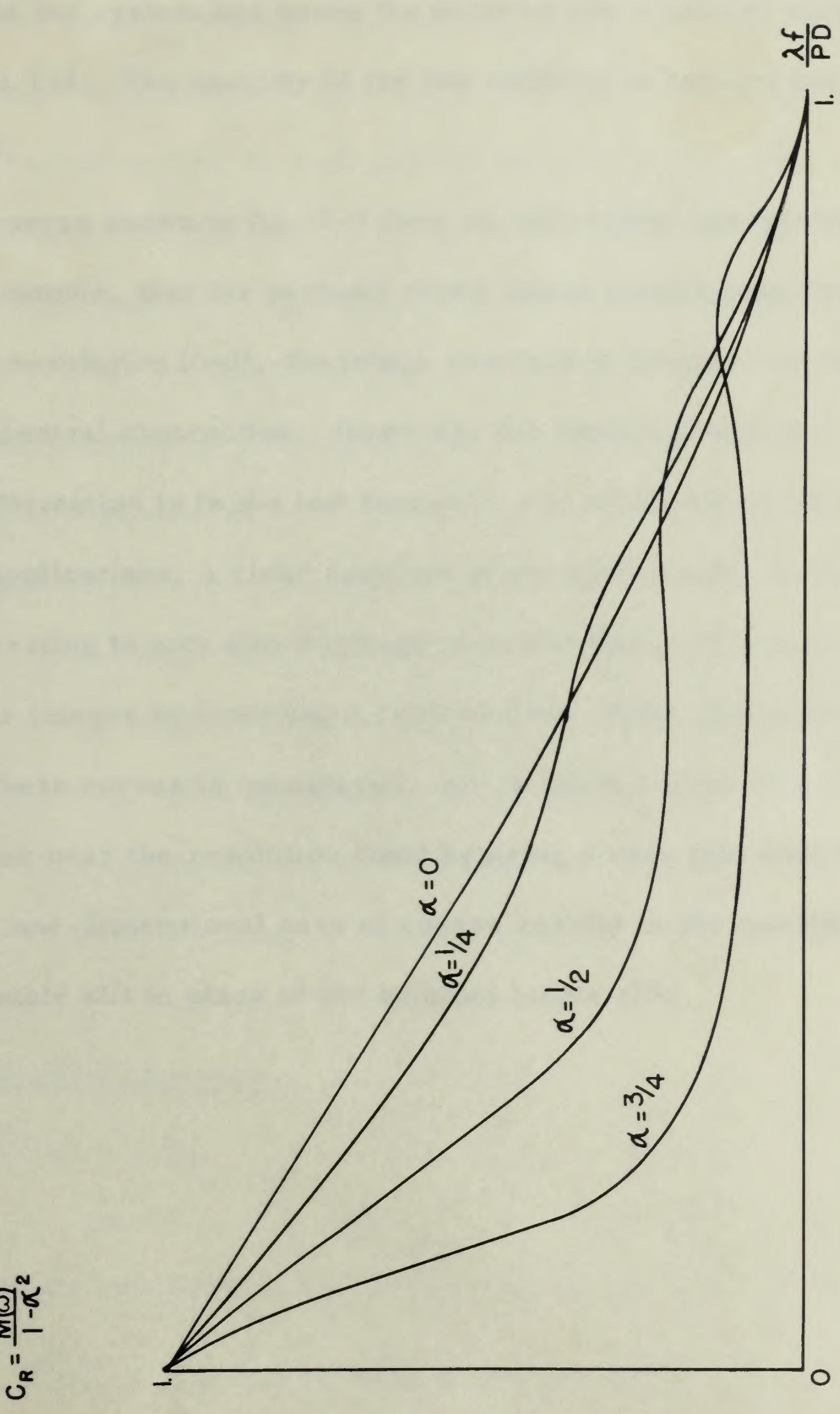

$C_R = \frac{M(\omega)}{1-\alpha^2}$
1.
0
1.
$\frac{\lambda f}{PD}$
$\alpha = 0$
$\alpha = 1/4$
$\alpha = 1/2$
$\alpha = 3/4$

Fig. 2-7

should be noted that originally the convolution integral was performed from the geometry of the system and hence the notation and variables differ from those of equ. 2.134. The equality of the two methods is brought out, however, in Appendix IV.

The curves shown in fig. 2-7 have an interesting interpretation. They suggest, for example, that for periodic detail whose fundamental frequency is close to the resolution limit, the image contrast is enhanced by the insertion of a central obstruction. However, for random detail for which most of the information is in the low frequency end of the spectrum as in photographic applications, a clear aperture is certainly more advisable.
It is also interesting to note that Rayleigh mentions Herschel's success in separating star images by inserting a central disk. If the geometrical scheme for obtaining these curves is considered, one is led to the obvious possibility of getting a peak near the resolution limit by using a very thin annular ring. The analogous one-dimensional case of course results in the interference pattern of a double slit in place of the original single slit.

III Scanning and Defocusing:

Thus far we have considered the problem of diffraction of plane waves by various shaped apertures and have found that there is an upper limit of resolution in terms of sine wave targets beyond which we are not able to separate two successive maxima. We further stated that, in theory at least, the method can be applied to any objects which are expandable in a Fourier series or in a Fourier Integral, and to any shaped aperture providing the expressions are integrable. The extension to random intensity distributions and more difficult aperture functions will be given in a later part of this paper.

For the present let us examine some ideal systems which are approximated in geometrical optics and television scanning. Perhaps the simplest case to deal with is that of a simple scanning aperture commonly employed in television systems (cf. fig. 3-1). This particular system lends itself to a simple analysis if we assume as an approximation that the response function $h(r)$ for a point source is simply a constant intensity over the circle and zero elsewhere as shown below.

A) Circular Scanning Aperture

3.1 $$h(r) = I_0 \qquad 0 \leq r \leq R$$
$$= 0 \qquad r > R$$

taking the two-dimensional Fourier transform with rotational symmetry

3.2 $$M(\omega) = \int_0^\infty h(r)\, J_0(\omega r)\, r\, dr$$

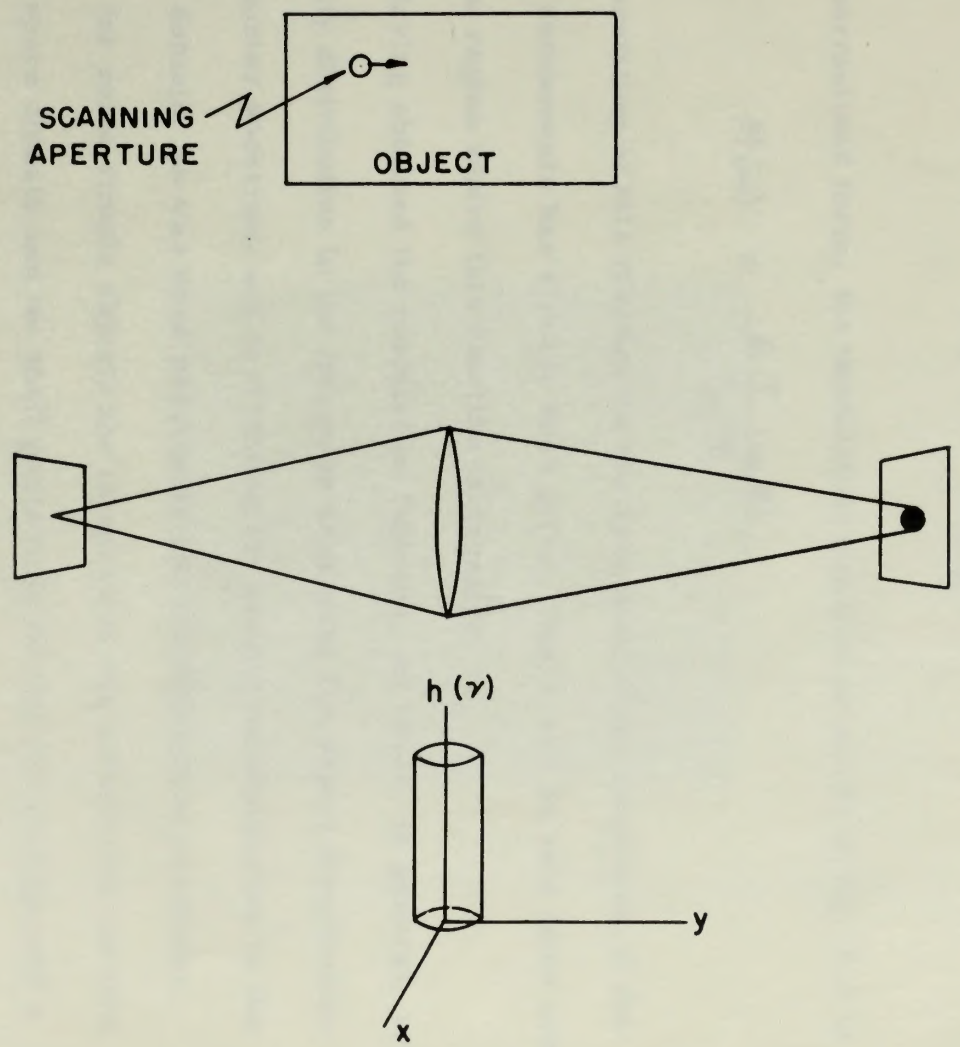

SCANNING
APERTURE
OBJECT
$h(\gamma)$
y
x

Fig. 3-1

which in our case becomes

3.3
$$M(\omega) = \int_0^R I_0 J_0 (\omega r) r \, dr$$

or

3.4
$$M(\omega) = I_0 R^2 \frac{J_1(\omega r)}{\omega R}$$

Thus, in normalized form, the modulation function as shown in fig. 3-2 is

3.5
$$M(\omega) = \frac{2 J_1 (\omega R)}{\omega R}$$

The interpretation of this function as the reduction of the amplitude of the sine wave components has already been given. More will be said later concerning the region where this function is negative.

Having obtained the modulation function, we could, in general, predict the intensity distribution in the image by expanding the object distribution into its Fourier spectrum and performing the proper multiplication in the frequency domain, as was done previously for the diffraction problems. However, for some simple objects the integration can actually be carried out in the space domain and we shall proceed to do this for an edge and a series of bars.

Before actually investigating the problem of scanning by a circular aperture or slit, let us first of all set up some general principles for problems of this sort. If the object to be scanned is a series of bars or an edge, the analysis can be simplified greatly without ever resorting to the frequency domain. Let us therefore formulate the problem in general where $f_{obj.}(x)$ is the intensity distribution of just one white bar on a black field.

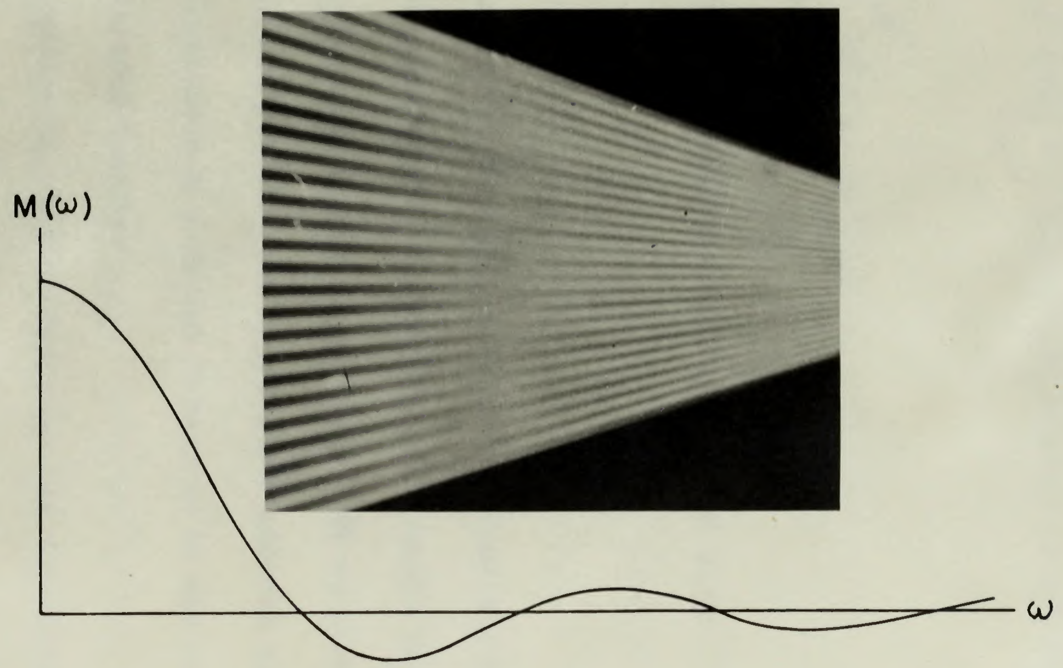


Fig. 3-2

Then

3.6 $$f_{image}(x') = \int_{-\infty}^{\infty} h(x-x')\, f_{obj.}(x)\, dx$$

and integrating by parts:

3.7 $$f_{image}(x') = f_{obj.}(x)\, I(x')\Big|_{-\infty}^{\infty} - \int I(x-x')\left[\delta(x+a) - \delta(x-a)\right] dx$$

where

3.8 $$I(x-x') = \int_{-\infty}^{x} h(x-x')\, dx$$

Since the first expression in (3.7) vanishes at the limits we have

3.9 $$f_{image}(x') = I(x'+a) - I(x'-a)$$

This is an extremely important relationship, i.e., we need merely know the response of the system to a point source and plot the area under this curve as a function of the variable. Then according to equation (3.9) we plot this distribution about each edge and perform the appropriate addition and subtraction graphically. The extension to a series of bars or to one edge is obvious and will be demonstrated shortly.

If the response of the system to a point source or line source is symmetric, then we can say the following:

that in the absence of spurious effects which will be discussed shortly

a) the distribution about an edge will be symmetric

b) the second derivative of the distribution vanishes at the edge

c) the intensity at the edge will be one half its' maximum value.

These properties hold until we arrive at the curious phenomena known as "spurious resolution", about which more shall be said later.

Applying these results to the problem of a circular scanning aperture we see that for a circle

3.10
$$I(X) = 2\int_{-R}^{X} \sqrt{R^2 - x^2}\, dx \qquad \eta = \frac{X}{R}$$

or in normalized form

3.11
$$I(X) = \begin{cases} 0 & \eta < -1 \\ \frac{2}{\pi}\left[\pi - \cos^{-1}\left(\frac{X}{R}\right) + \left(\frac{X}{R}\right)\sqrt{1-\left(\frac{X}{R}\right)^2}\right] & -1 \leq \eta \leq 1 \\ 1 & \eta > 1 \end{cases}$$

Thus whether the target is a single edge or a series of bars we merely plot equation (3 11) about each edge and add up the contributions graphically with the appropriate signs. If we do this for various values of the ratio of circle radius to bar width we find that the maximum intensity decreases until the distribution is practically constant (cf. fig. 3-2), whence as the circle radius increases with respect to the width, the maxima begins to build up again in between the pulses, i.e., the original black and white bars have reversed their positions. This 180° phase shift in space is called "spurious resolution".

For a general treatment of this phenomena, it is simpler at first to consider a sine wave target and then apply the results to the bar (or square wave) target. The blurred image of a sine wave target is given by

3.12
$$I_m(x) = B_0 + M(\omega) B_1 \sin \omega x$$

and in the case that we are interested in presently, namely, scanning with a circular aperture, $M(\omega)$ is given in fig 3-2 . Noting that the variable is

3.13
$$\omega R = 2\pi\left(\frac{R}{P}\right)$$

which is a measure of the ratio of the circle radius to the period of the sine wave, we see that as this variable increases the amplitude of the sine wave is continuously decreased and actually vanishes when

3.14
$$\omega R = \frac{2\pi R}{P} = 3.83$$

3.83 being the first root of $J_1(x)$

For this particular value, the blurred image reduces to a constant distribution since $M(\omega) = 0$ in (equation 3.12).

Further increase in the variable $\frac{R}{P}$ shows that $M(\omega)$ becomes negative, whence 3.12 becomes

3.15
$$I_m(x) = B_0' - |M(\omega)| \, B_1 \sin \omega x$$

or a 180° phase shift in space has occurred. This amplitude modulation and spatial phase shift will continue to occur according to fig. 3-2.

Let us now apply the previous analysis to a square wave or bar target. Its blurred image distribution is given by

3.16
$$I_m(x) = B_0' + M(\omega) C_1 \sin \omega x + M(3\omega) C_3 \sin 3\omega x + \cdots$$

$$C_n = \frac{4}{n\pi} \qquad n \text{ odd}$$

From the relative magnitude of the harmonic terms we see that the fundamental is predominant and therefore let us describe the distribution in (equation 3.16) as $\frac{R}{P}$ increases. Here again we see that the amplitude of the fundamental and the upper harmonics are continuously decreased until, when the conditions

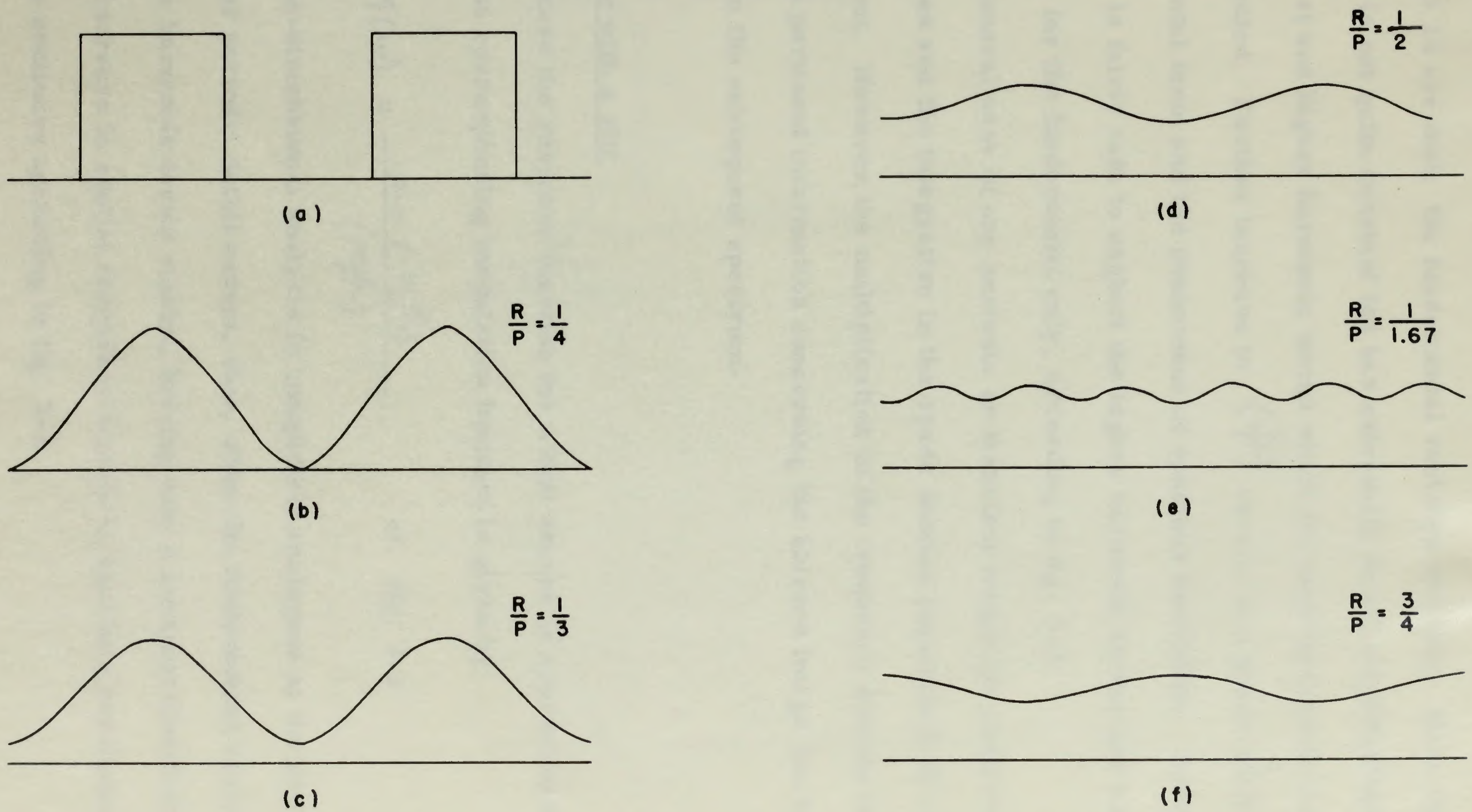

(a)
$\frac{R}{P} = \frac{1}{4}$
(b)
$\frac{R}{P} = \frac{1}{3}$
(c)
$\frac{R}{P} = \frac{1}{2}$
(d)
$\frac{R}{P} = \frac{1}{1.67}$
(e)
$\frac{R}{P} = \frac{3}{4}$
(f)

Fig. 3-3

in equation 3.14 are met; the fundamental vanishes entirely. Here the distribution is not quite constant but has extremely small contributions from the first and higher harmonic terms which for most practical cases can be neglected. Further increase in $\left(\frac{R}{P}\right)$ results in a phase shift of the fundamental term and the phenomena of spurious resolution. Beyond this point it is fairly safe to neglect the higher harmonic terms and continue the analysis for the fundamental only, according to fig. 3-3.

In the general case of any periodic or transient target the analysis becomes more complex and the integration in the space domain (equation 3.6) cannot be carried out. However, the multiplication in the frequency domain is still possible and pertinent information concerning the blurred image can be derived from the subsequent spectrum.

B. Scanning with a slit

In this case the response function for a line source is approximated by a pulse. The corresponding modulation function is given by

3.17
$$M(\omega) = \frac{\sin\left(\frac{\omega d}{2}\right)}{\left(\frac{\omega d}{2}\right)} \qquad \text{cf. fig. 3-4}$$

This one-dimensional analysis is completely analogous to the previous treatment for periodic detail except, that, when the fundamental vanishes, all the upper harmonic terms vanish, leaving only a constant distribution. Subsequent increase in spatial frequency results in spurious resolution and this process continues according to fig. 3-4.

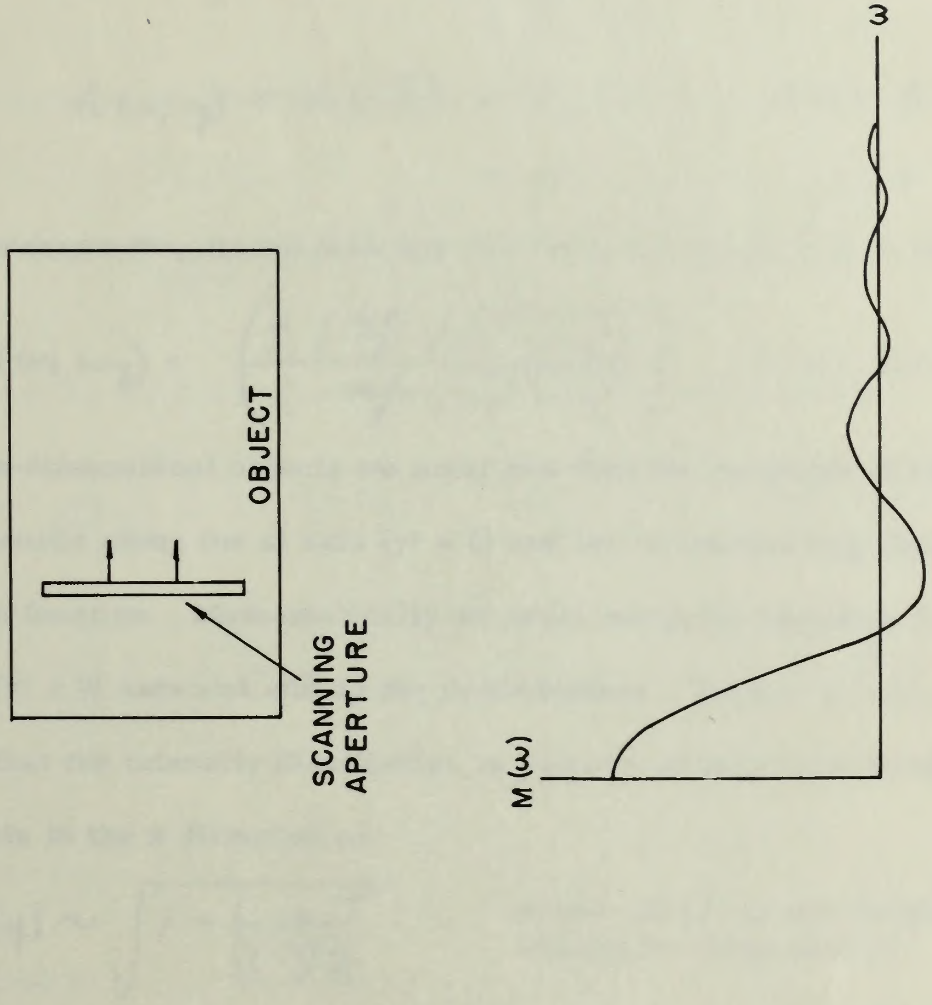

OBJECT
SCANNING APERTURE
M(ω)
ω

Fig. 3-4

C. Ideal Defocused Lens:

For this idealized problem, one must essentially expand the object into its two-dimensional Fourier spectrum and obtain the corresponding modulation function in order to arrive at the image spectrum. For the ideal lens out of focus we have as an approximation for the intensity distribution fig 3-1.

3.18
$$h(x, y) = h(\vec{r}) = I_0 \qquad 0 \le r \le \frac{aR}{f}$$
$$= 0 \qquad r > \frac{aR}{f}$$

and the corresponding modulation function from equations 3. 3, 3. 4, 3. 5 is

3.19
$$M(\omega_x, \omega_y) = \left(\frac{J_1\left(\frac{aR}{f}\sqrt{\omega_x^2 + \omega_y^2}\right)}{\frac{aR}{f}\sqrt{\omega_x^2 + \omega_y^2}}\right)$$
cf. fig. 3-2

For one-dimensional objects we must now find the response of the system to a line source along the x' axis (y' = 0) and the corresponding one-dimensional modulation function Mathematically we must integrate (equation 3. 18) along the (y' = 0) axis and add up the contributions. Suffice to say, it can be shown that the intensity distribution is linearly proportional to the chord of the circle in the x direction or

3. 20
$$l(y) \sim \sqrt{1 - \left[\frac{y}{\left(\frac{aR}{f}\right)}\right]^2}$$
where $l(y)$ is the response of the system to allied source

However, we can arrive at this result and demonstrate the convenience of working in the frequency domain at the same time, so let us make use of the relationship:

3. 21
$$F_{image}(\omega_x, \omega_y) = M(\omega_x, \omega_y)\, F_{obj.}(\omega_x, \omega_y)$$

where $M(\omega_x, \omega_y)$ is given in equation 3.19 and since the object in this case is a line source along the x' axis, i.e., $\delta(y')$ about the origin on the y' axis, the spectrum of which is a constant along ω_y and zero elsewhere; we have

3.22
$$F_{image}(\omega_y) = \frac{J_1\left(\frac{aR}{f}\omega_y\right)}{\left(\frac{aR}{f}\omega_y\right)}$$

The corresponding image distribution is the inverse Fourier transform of 3.22, i.e.,

3.23
$$\ell(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{J_1\left(\frac{aR}{f}\omega_y\right)}{\left(\frac{aR}{f}\omega_y\right)}e^{i\omega_y y}\,d\omega_y$$

or making use of the integral representation for Bessel functions

3.24
$$\ell(y) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}\frac{1}{2}J_0\left(\frac{aR}{f}\omega_y\right)e^{i\omega_y y}\,d\omega_y + \int_{-\infty}^{\infty}\frac{1}{2}J_2\left(\frac{aR}{f}\omega_y\right)e^{i\omega_y y}\,d\omega_y\right]$$

i.e.

3.25
$$\ell(y) = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega_y y}\,d\omega_y\int_{0}^{\pi}e^{-i\frac{aR}{f}\omega_y\sin\theta} + e^{i\left(2\theta - \frac{aR}{f}\omega_y\sin\theta\right)}\,d\theta\right]$$

3.26
$$\ell(y) = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}e^{i\omega_y y}\,d\omega_y\int_{0}^{\pi}\cos\theta\, e^{-i\frac{aR}{f}\omega_y\sin\theta}\cos\theta\,d\theta\right]$$

Now let us define

3.27
$$\eta = \frac{aR}{f}\sin\theta \qquad \cos\theta = \sqrt{1-\left[\frac{\eta}{\left(\frac{aR}{f}\right)}\right]^2}$$

so 3.26 becomes

3.28
$$\ell(y) = \frac{1}{\sqrt{2\pi}}\left[\frac{f}{\pi aR}\int_{-\infty}^{\infty}e^{i(y-\eta)\omega_y}\,d\omega_y\int_{-\frac{aR}{f}}^{\frac{aR}{f}}\sqrt{1-\left[\frac{\eta}{\left(\frac{aR}{f}\right)}\right]^2}\,\right]d\eta$$

Collecting all constant terms and integrating first over ω_y , we have

3.29
$$\ell(y) = C\int_{-\frac{aR}{f}}^{\frac{aR}{f}} \sqrt{1-\left[\frac{\eta}{\frac{aR}{f}}\right]^2}\;\delta(y-\eta)\,d\eta$$

whence normalized 3.29 becomes

3.30
$$\ell(y) = \sqrt{1-\left[y/\left(\frac{aR}{f}\right)\right]^2} \qquad 0 \le y \le \frac{aR}{f}$$

Hence 3.30 represents the response of the system to a line source and the corresponding one-dimensional modulation function is

3.31
$$M(\omega_y) = \frac{J_1\left(\frac{aR}{f}\,\omega_y\right)}{\left(\frac{aR}{f}\,\omega_y\right)}$$

We thus see by fig. 3-2 that again spurious resolution occurs for periodic detail and the subsequent analysis is similar to that of a circular scanning aperture outlined previously.

While the response functions employed for the scanning apertures may give satisfactory results, the analogous treatment for defocused optical systems is certainly too ideal to give quantitative information regarding the systems' behavior. In the more realistic cases, however, we can still make a few general remarks concerning the image deterioration.

1) If the response of the system to a line source is $\ell(y)$ then we can construct the image of an edge or a series of bars by plotting $I(y)$ as defined in equation 3.8 about each edge and add up the contributions graphically (equation 3.9).

2) Conversely the quality of a lens system is often measured in terms of how an edge is deteriorated. From equation 3.8 we see that the response of the system to a line source is the derivative of the edge deterioration function.

3) If the response of the system to a line source cannot be derived analytically it might be possible to fit a curve to the experimental response or to the slope of the edge function, whence a Fourier transformation will give the corresponding modulation function.

4) For periodic sine wave and bar targets in the laboratory, and for some random distributions, the analysis is greatly facilitated by knowing the modulation function and performing the analysis in the frequency domain. However, for the case where a single or several edges are involved it is more informative to perform the analysis by means of equations 3.6, 3.7, 3.8, 3.9.

D. The Synthesis Problem

Up to this point we have concerned ourselves with an examination of what might be termed ideal systems from physical and geometric optics. Within the approximation of linearity, as previously defined, it was found that there existed a direct analogy between the optical systems and electrical network theory. For the analysis of such systems in general, one must know the system function or modulation function in order to carry on the analysis in the frequency domain. This implicitly demands a knowledge of the systems response to a unit impulse and the capability of taking its Fourier transforms.

Most of the attention thus far has been focused on the evaluation of the modulation function and its interpretation. While only a few cases have been cited a large number of systems occur to which this scheme is applicable.

A general treatment on the extension of the method to non-ideal systems will be given in a latter portion of the paper.

The natural step to take at this point, in keeping with the electrical analogy, is to design an optical system to do a given job; i.e. the design of an optimum system.

Here again the format for designing such a system has already been established in electrical network theory. Its optical counterpart has been brought out by Elias, Robinson & Grey (ref. #5). In general the problem reduces to one for which the actual input and the desired output are known, and from this one evaluates the modulation or system function. By taking the Fourier transform of the ratio of output/input spectra one obtains the desired response function. One then wishes to devise a system to give this response function.

Before proceeding further it seems advisable to clarify the issue with a brief discussion of random intensity distributions. For simplicity we shall deal only with one dimensional functions, tacitly assuming that the results are extendable to two or n-dimensions as has been shown in refs. 5 and 3.

Random Functions

Assume that we are given a random intensity distribution $f_{obj.}(x)$ extending from $-\infty$ to $+\infty$. For such a function a Fourier transform does not exist since $\int_{-\infty}^{\infty} | f_{obj.}(x) | \, dx$ is infinite, violating the condition mentioned in sec. I. Also, by employing the "complex multiplication theorem" one can show that

3.32a $$\int_{-\infty}^{\infty} |g(\omega)|^2 e^{i\omega x}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, f(x+x_0)\, dx$$

is no longer valid for the same reason. However, if $\lim_{P\to\infty} \frac{1}{2P}\int_{-P}^{P} |f(x)|^2\, dx$ exists, then both sides of equation 3.32a can be made finite by dividing each side by a large period, i.e., taking a space average.

Defining the "power density spectrum" as

3.32b $$\Phi_{11}(\omega) = \lim_{P\to\infty} 2\pi \frac{|g(\omega)|^2}{2P}$$

then 3.32a becomes

3.33 $$\int_{-\infty}^{\infty} \Phi_{11}(\omega) e^{i\omega x}\, d\omega = \lim_{P\to\infty} \frac{1}{2P}\int_{-P}^{P} f(x)\, f(x+x_0)\, dx$$

where the right hand member is the auto-correlation function of $f(x)$, namely, $\varphi_{11}(x_0)$. We see, therefore, that the power density spectrum and auto-correlation function are Fourier transform pairs.

Further considerations (cf. ref. #2) lead to the important relation that

3.34 $$\Phi_{image}(\omega) = |M(\omega)|^2\, \Phi_{obj.}(\omega)$$

which illustrates that the methods previously established for the synthesis and analysis of linear systems are still applicable even when dealing with random functions..

The derivation of the auto-correlation function from Mathematical Statistics and the subsequent demonstration of its Fourier relationship with the power density spectrum provided the communications engineer with a

AUTO-CORRELATION FUNCTION

$$\phi_{11}(x,y) = \lim_{p \to \infty} \frac{1}{p^2} \int_{-\frac{p}{2}}^{p/2} \int_{-\frac{p}{2}}^{p/2} f(r,s)\, f(r+x,s+y)\, dr\, ds$$

POWER DENSITY SPECTRUM

$$\Phi_{11}(\omega_x,\omega_y) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \phi_{11}(x,y)\, e^{-i(\omega_x x+\omega_y y)}\, dx\, dy$$

$$\Phi_{\text{image}}(\omega x,\omega y) = \left| M(\omega x,\omega y) \right|^2 \Phi_{\text{object}}(\omega x,\omega y)$$

Fig. 3-5

powerful tool for the synthesis and analysis of electrical networks whose inputs are of a random nature. A comprehensive treatment of this subject is not within the province of this paper, and is presented here in outline form only. For a clear and comprehensive treatment of the matter, along with the design of an "optimum linear system" in general, the reader is urged to read (ref. 2). The extension of the foregoing to two or more dimensions is straightforward and can be found in refs. 3 and 5. An experimental means of evaluating auto and cross correlation functions is discussed in ref. 18. With a method of dealing with random distributions, it appears that an optical system can be designed to give optimum performance from the Wiener-Hopf criterion of least mean square error (#2). Suffice to say here that from the auto and cross correlation functions of the desired and undesired portions of the object and image intensity distributions, it is possible to evaluate a modulation function which will optimize the response of the system.

This is of course a filtering process, and the problem is thus one of designing a system whose point source response is the Fourier Transform of the modulation function. To this end therefore, the system shown in fig. 3-6 was devised. The general scheme involves the evaluation of $h(x, y)$ the point source response function from the desired modulation function. By inserting a variable transmission mask in the stop of the lens whose transmission varies according to $h(x, y)$, and viewing the image in an out-of-focus setting, we insure that every point in the object gives a distribution $h(x, y)$ in the defocused image plane.

The first attempt was to design a sharp cut-off low pass filter as shown

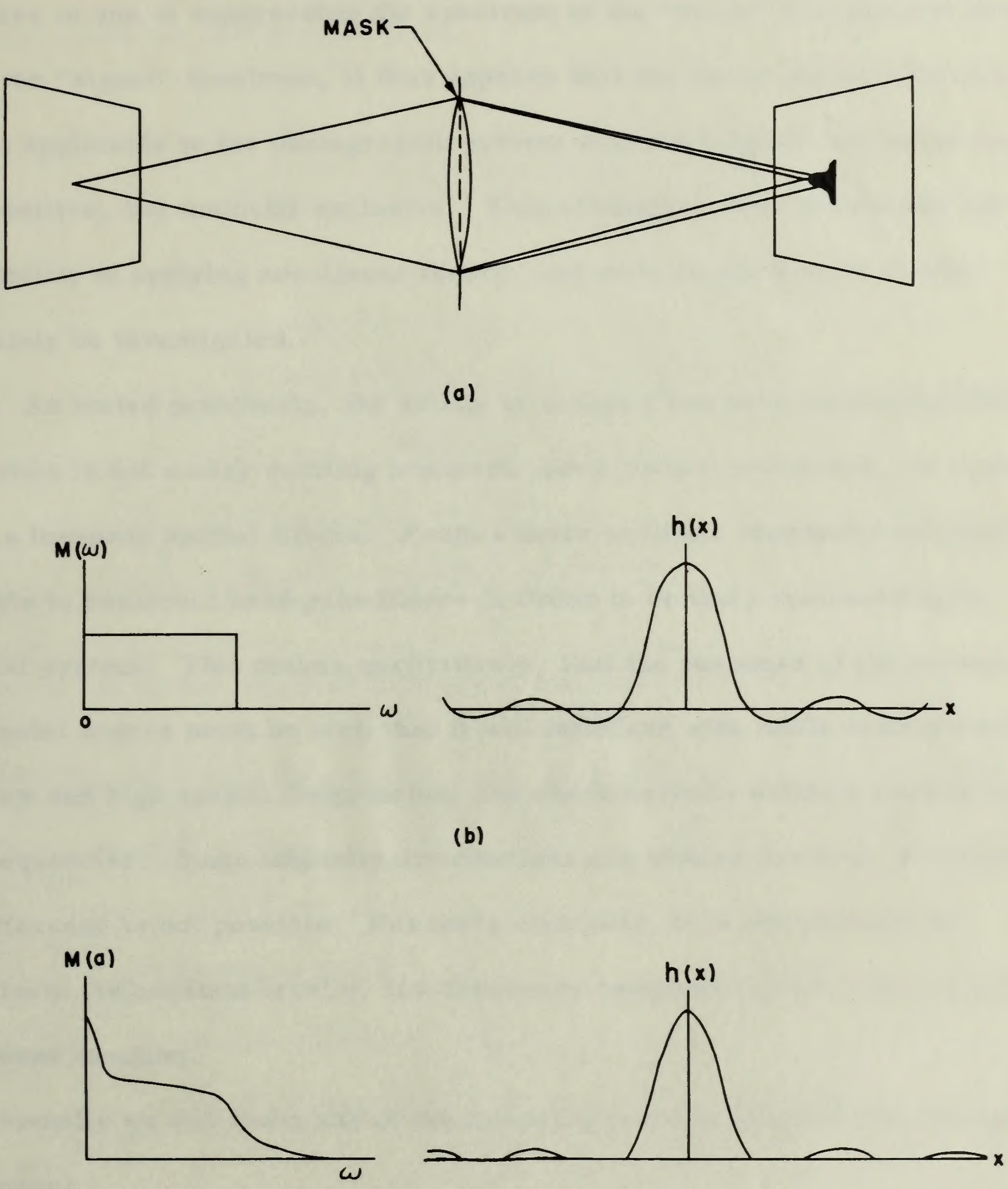
in (fig. 3-6b) with the corresponding response function $h(x)$. However, noting that $h(x)$ is an intensity distribution, i.e. $h(x) \geq 0$ for all x , it was decided to neglect the negative portion of the function (as a first approximation) and determine the modified modulation function, noting in the process that $h(x)$ is not infinite in extent but localized in space. The modified response function and modulation function are shown in fig. 3-6c. A color coded system using two T V pick up tubes has been suggested by Dr. A. Kohlenberg and is discussed in ref. 3 to overcome the restrictions placed on $h(x)$.

From the standpoint of practicality, the production of low-pass spatial filters in optics is hardly noteworthy since all optical systems behave as low-pass filters. More stimulating surely is the possibility of producing band-pass modulation filters. After several attempts were made in this direction it was decided to scrutinize more thoroughly the basic problems involved in spatial filtering, keeping in mind the ultimate goal for which the filters were to be used.

As a result of this investigation the following list of restricting limitations was placed on the process of spatial filtering.

Limitations:

1) Since, in a photograph, the desirable and undesirable portions are not additive (as signal and noise in electrical theory); but rather mutually exclusive, both will contain essentially the same spectral distribution. This is particularly clear in a photograph of large detail, super-imposed on a


MASK
M(ω)
0
ω
h(x)
x
M(a)
ω
h(x)
x

(a)

(b)

(c)

Fig. 3-6

background of random fluctuations. Since, basically, the optimization problem reduces to one of suppressing the spectrum of the "noise" to a greater extent than the "signal" spectrum, it thus appears that the linear optimization process is not applicable to the photographic system wherein "signal" and noise are not additive, but mutually exclusive. This, of course, does not exclude the possibility of applying non-linear theory, and such an application should certainly be investigated.

2) As stated previously, the ability to design a low pass modulation filter in Optics is not a very exciting prospect, since optical processes, as a whole act as low-pass spatial filters. From a more realistic standpoint one must be able to construct band-pass filters in order to be truly synthesizing an optical system. This means, qualitatively, that the response of the system to a point source must be such that it will interfere with itself destructively for low and high spatial frequencies, and constructively within a certain band of frequencies. Since intensity distributions are always positive, destructive interference is not possible. Put more concisely, it is not possible to eliminate the constant level or low-frequency components with a non-negative response function.

Formally we can make use of the following proof to establish the foregoing statement.

Given that $f(x) \geq 0$ for all x

and $f(x) \rightarrow 0$ as $x \rightarrow \infty$

then we shall show that $|g(\omega)| \leq |g(0)|$ i.e., the response is a maximum for $\omega = 0$. It is important to note that for all practical response

functions, $h(x)$ is an even function so that we need merely concern ourselves with the cosine transform.

Hence:

3.35
$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

or

3.36
$$g(\omega) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} f(x) \cos \omega x\, dx$$

and

3.37
$$g(0) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(x)\, dx$$

We can break up the region of integration into several parts, namely

3.38
$$g(\omega) = \frac{2}{\sqrt{2\pi}} \left\{ \int_{0}^{\frac{\pi}{2\omega}} f(x) \cos \omega x\, dx + \sum_{n\ \mathrm{even}} \int_{\frac{(2n-1)\pi}{2\omega}}^{\frac{(2n+1)\pi}{2\omega}} f(x) \cos \omega x\, dx + \sum_{n\ \mathrm{odd}} \int_{\frac{(2n-1)\pi}{2\omega}}^{\frac{(2n+1)\pi}{2\omega}} f(x) \cos \omega x\, dx \right.$$

such that:

3.39
$$\int_{\frac{(2n-1)\pi}{2\omega}}^{\frac{(2n+1)\pi}{2\omega}} f(x) \cos \omega x\, dx \geq 0 \qquad \text{for n even}$$

3.40
$$\int_{\frac{(2n-1)\pi}{2\omega}}^{\frac{(2n+1)\pi}{2\omega}} f(x) \cos \omega x\, dx \leq 0 \qquad \text{for n odd}$$

Also keeping in mind that: $|\cos \omega x| \leq 1$

we have:

3.41
$$\sum_{n\ \mathrm{even}} \left| \int_{\frac{(2n-1)\pi}{2\omega}}^{\frac{(2n+1)\pi}{2\omega}} f(x) \cos \omega x\, dx \right| \leq \sum_{n\ \mathrm{even}} \left| \int_{\frac{(2n-1)\pi}{2\omega}}^{\frac{(2n+1)\pi}{2\omega}} f(x)\, dx \right|$$

and

3.42 $$\sum_{n\ \text{odd}} \left| \int_{\frac{(2n-1)\pi}{2\omega}}^{\frac{(2n+1)\pi}{2\omega}} f(x) \cos \omega x \, dx \right| \leq \sum_{n\ \text{odd}} \left| \int_{\frac{(2n-1)\pi}{2\omega}}^{\frac{(2n+1)\pi}{2\omega}} f(x) \, dx \right|$$

Now from equation 3.37 for $g(0)$, which in turn could be broken up into a series of such integrals

3.43 $$|g(\omega)| \leq |g(0)|$$

If we now identify $h(x)$ with $f(x)$ and $M(\omega)$ with $g(\omega)$, we see that the maximum response of a system whose filter function is non-negative is at the origin in the frequency domain as we should expect.

A more practical way of looking at this is to consider what happens in the optical and photographic systems with regard to edges. On the practical assumption that the point image distribution is much smaller than the spacing between edges, then it follows that each edge will be rounded off in the same fashion whether it be the edge of a long thin object or a short bulky object. At best one could create a relatively large circle of confusion to filter against closely spaced objects. This limitation is not strictly enforced when dealing with periodic detail, however, where it may be possible to pass only a certain band of spatial frequencies. An interesting example of this would be to view a variable sine wave target, with a mask in the stop of the lens whose transmission varies sinusoidally with a frequency corresponding, for example, to the mean frequency of the target.

In the general case of random detail, however, it is the author's opinion that the insertion of a variable transmission mask together with the defocused image plane merely creates an artificial circle of confusion which deteriorates

the edge of a long flat building in the same proportion as the edge of a tall thin skyscraper.

Recommendations:

Despite the restrictions cited above when dealing with transmission masks and geometrical optics, there nevertheless appears to be several cases where the synthesis procedure is still applicable. For example, on the basis of the assumptions made in geometrical optics, one must conclude that for an ideal system the optimum focal setting is the Gaussion focal plane where the point image distribution is theoretically a point. However, upon closer examination one finds that instead of a point, the image distribution is the conventional diffraction pattern. One may then ask, "Is it possible to redistribute the energy in the diffraction pattern for specific applications?" From the Fourier relation between the amplitude in the image plane and the amplitude and phase function in the exit pupil (cf. Sec. II), the answer is in the affirmative. Such a synthesis procedure is called "apodization" and has been carried on in France in recent years by Jacquinot.

In addition to this, of course, is the method of sharpening edges by electronic means and the process of photographic masking as outlined by Otto Schade. It is of interest to note that the main difficulty encountered in large scale spatial filtering is due to the fact that incoherent illumination is employed. Such restrictions are not applicable when dealing with coherent or partially coherent sources. For such applications the reader is referred to ref. #17 and the subsequent references cited therein.

Thus, in the realm of physical optics, a logical extension of the material presented here would seem to lie in the work of Marechal who has evaluated modulation functions for non-ideal systems, i.e., systems having a phase distribution over the exit pupil. In this regard Nijboer in his dissertation (ref. 10) has expanded an arbitrary phase distribution into a series of orthogonal functions and has associated with each of the first few terms the various aberration coefficients. It would indeed be interesting to evaluate the effect of the various aberrations on contrast reduction by one of the several analytic or experimental schemes illustrated in the section on diffraction theory. As an aside, it should be mentioned that the Fourier method of dealing with diffraction theory is not limited to the visual portion of the electromagnetic spectrum. Recent literature from England indicates that the Fourier transform relationship between the amplitude distribution and the phase function over an aperture has proven to be a powerful tool in three dimensions for analyzing spacings and structure on an atomic and molecular scale in X-ray diffraction theory.

Restricting ourselves to incoherent sources and Gaussian optics, it is now felt that there are still many quasi-ideal problems subject to the analysis presented in this paper, e. g., object and/or image motion, vibrational problems in shutter design, etc. An investigation should be made to include aberrational effects for sine wave, square wave targets or edge testing in the laboratory.

As for synthesizing an optical system, in general, in terms of optimization, it is felt that the limitations previously mentioned place such stringent restrictions on the designer as to be impractical except perhaps in a few, well chosen,

isolated cases. Also, the two-dimensional problem, without radial symmetry, dealing with the whole field of the object would appear to be most forbidding.

At this point it must be emphasized that the pessimistic attitude predominant in the above paragraph is confined to systems which are entirely optical in nature. By this we mean that if one wishes to synthesize a photograph, the flexibility which an optical process alone affords is quite limited. However, by transducing the spatial frequencies into one-dimensional electronic signals, one can perform filtering operations which will enhance the final output. Schemes such as sharpening edges and the like in blurred images are already in progress in the field of electronics. It has been argued, however, that perhaps the insertion of a low-pass spatial frequency filter in the optical system would be at times advisable. By way of illustration, one could consider a white cardboard with confetti scattered over it at random, but in such a way that there exists the diffuse outline of some numeric which is not particularly clear to the eye in sharp focus. Viewing this object through a piece of ground glass, the subject easily identifies the numeric. Actually the random fluctuations in the background have been averaged out to a degree. Using a lens system, there appears to be no need of inserting a variable transmission mask as a filter to identify the number when merely defocusing slightly or scanning the object would give the same result, since these processes, as we have seen, behave inherently as low-pass filters.

SUMMARY

It has been the purpose of this paper to present a method for dealing with optical processes in terms of their capability for preserving object detail in the final image. From electrical network theory an analogy was established by means of which one can actually evaluate a function, the modulation function, which measures contrast rendition as a function of the spatial frequencies. Several ideal cases arising from physical and geometric optics were considered and emphasis was placed on the several schemes for evaluating the corresponding modulation functions. Extending the analogy between the optical process and electrical network theory to the problem of synthesis, it was felt that if one is restricted to an optical system only, the general synthesis procedure is too limited to be practical. By including electronic devices and photographic techniques it is possible to enhance the final image and perform practical spatial filtering. Finally, it is the author's opinion that, despite the restrictions on the synthesis problem, the Fourier method is unparalleled in its approach to the analysis of optical systems in terms of performance.

Appendix 2-I

$$M(\omega) = \frac{2}{\omega_0}\int_0^\infty \left(\frac{\sin\eta}{\eta}\right)^2 \cos\alpha\eta \, d\eta \qquad \text{where} \quad \begin{array}{l} \eta = \omega_0 y \\ \alpha = \frac{\omega}{\omega_0} \end{array}$$

$$= \frac{2}{\omega_0}\int_0^\infty \frac{\sin\eta}{\eta^2} \sin\eta \cos\alpha\eta \, d\eta$$

$$= \frac{1}{\omega_0}\int_0^\infty \frac{\sin\eta}{\eta^2} \sin(1+\alpha)\eta \, d\eta + \frac{1}{\omega_0}\int_0^\infty \frac{\sin\eta}{\eta^2} \sin(1-\alpha)\eta \, d\eta$$

$$= \frac{I_1}{\omega_0} + \frac{I_2}{\omega_0}$$

Consider:

$$I_1 = \int_0^\infty \frac{\sin\eta \sin(1+\alpha)\eta}{\eta^2} \, d\eta$$

and integrate by parts letting

$$u = \sin\eta \sin(1+\alpha)\eta$$
$$dv = \frac{d\eta}{\eta^2}$$

We have:

$$I_1 = \left.\frac{-\sin\eta \sin(1+\alpha)\eta}{\eta}\right|_0^\infty + \int_0^\infty \frac{\cos\eta \sin(1+\alpha)\eta}{\eta} \, d\eta + \int_0^\infty \frac{(1+\alpha)\sin\eta \cos(1+\alpha)\eta}{\eta} \, d\eta$$

which can be written as

$$I_1 = \int_0^\infty \left[\frac{\cos\eta \sin(1+\alpha)\eta + \sin\eta \cos(1+\alpha)\eta}{\eta}\right] d\eta + \alpha\int_0^\infty \frac{\sin\eta \cos(1+\alpha)\eta}{\eta} \, d\eta$$

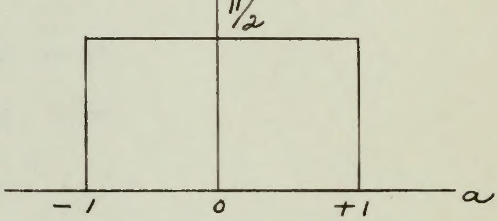
or

$$I_1 = \int_0^\infty \frac{\sin(\alpha+2)\eta}{\eta}\,d\eta + \alpha \int_0^\infty \frac{\sin\eta\,\cos(1+\alpha)\eta}{\eta}\,d\eta$$

$$I_1 = I_3 + \alpha I_4$$

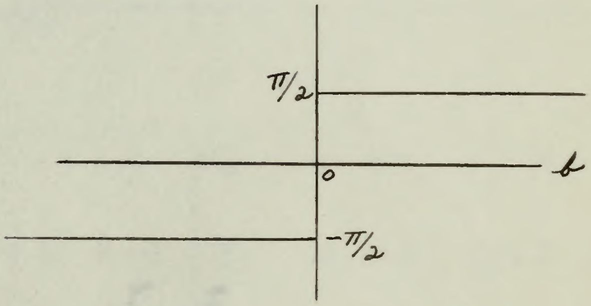
and noting that

$$\int_0^\infty \frac{\sin x\,\cos a x}{x}\,dx = \begin{cases} 0 & a<-1;\ a>1 \\ \pi/4 & a=\pm 1 \\ \pi/2 & a^2<1 \end{cases}$$



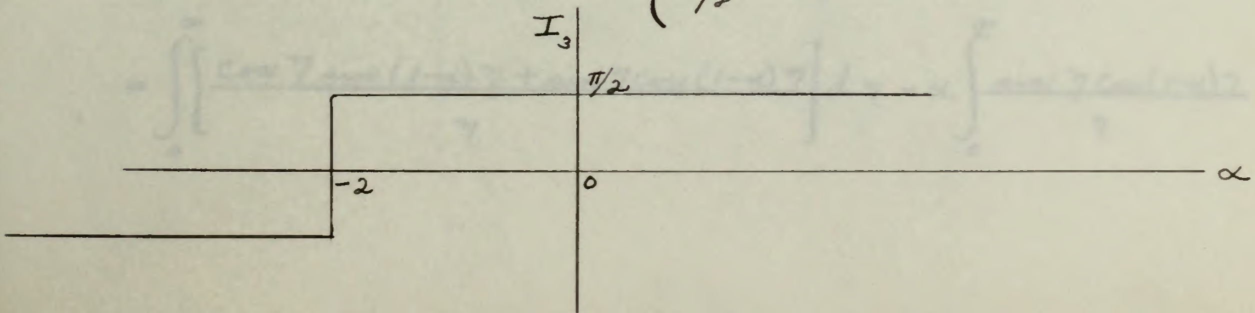
and that

$$\int_0^\infty \frac{\sin b x}{x}\,dx = \begin{cases} \pi/2 & b>0 \\ 0 & b=0 \\ -\pi/2 & b<0 \end{cases}$$



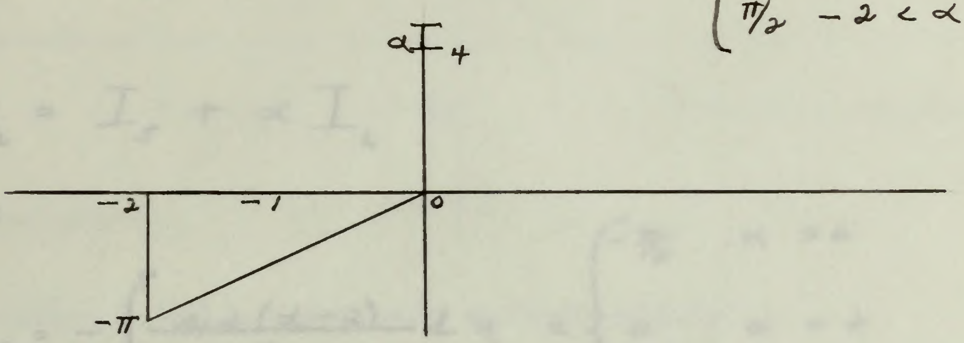
we have:

$$I_3 = \int_0^\infty \frac{\sin(2+\alpha)\eta}{\eta}\,d\eta = \begin{cases} \pi/2 & \alpha>-2 \\ 0 & \alpha=-2 \\ -\pi/2 & \alpha<-2 \end{cases}$$



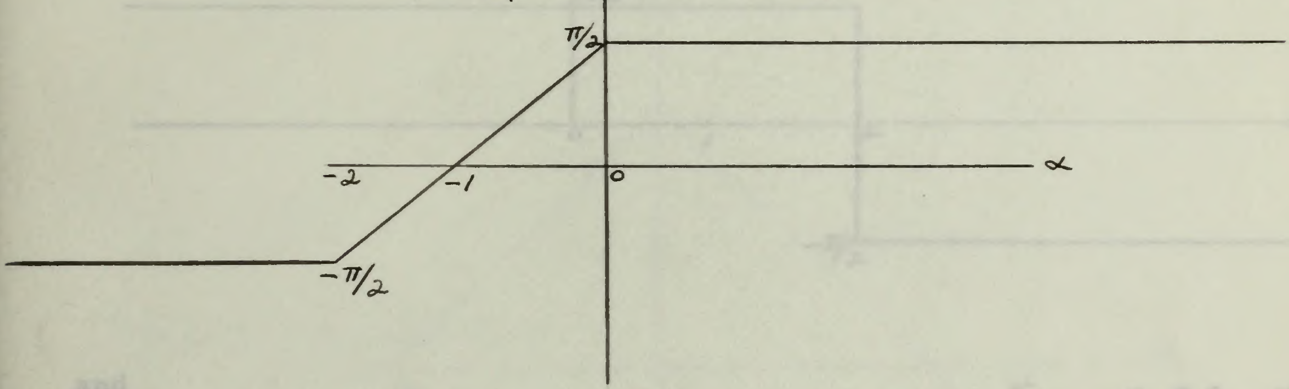
also

$$\alpha I_4 = \alpha\int_0^\infty \frac{\sin\eta\cos(2+\alpha)\eta}{\eta}\,d\eta = \alpha\begin{cases}0 & \alpha<-2,\ \alpha>0\\ \pi/4 & \alpha=-2,\ 0\\ \pi/2 & -2<\alpha<0\end{cases}$$



the addition of which gives:

$$I_1 = I_3 + \alpha I_4$$



and in a similar fashion:

$$I_2 = \int_0^\infty \frac{\sin\eta\sin(1-\alpha)\eta}{\eta^2}\,d\eta = -\frac{\sin\eta\sin(1-\alpha)\eta}{\eta}\Bigg|_0^\infty + \int_0^\infty \frac{\cos\eta\sin(1-\alpha)\eta}{\eta}\,d\eta$$

$$+(1-\alpha)\int_0^\infty \frac{\sin\eta\cos(1-\alpha)\eta}{\eta}\,d\eta$$

$$= \int_0^\infty\left[\frac{\cos\eta\sin(1-\alpha)\eta+\sin\eta\cos(1-\alpha)\eta}{\eta}\right]d\eta - \alpha\int_0^\infty \frac{\sin\eta\cos(1-\alpha)\eta}{\eta}\,d\eta$$

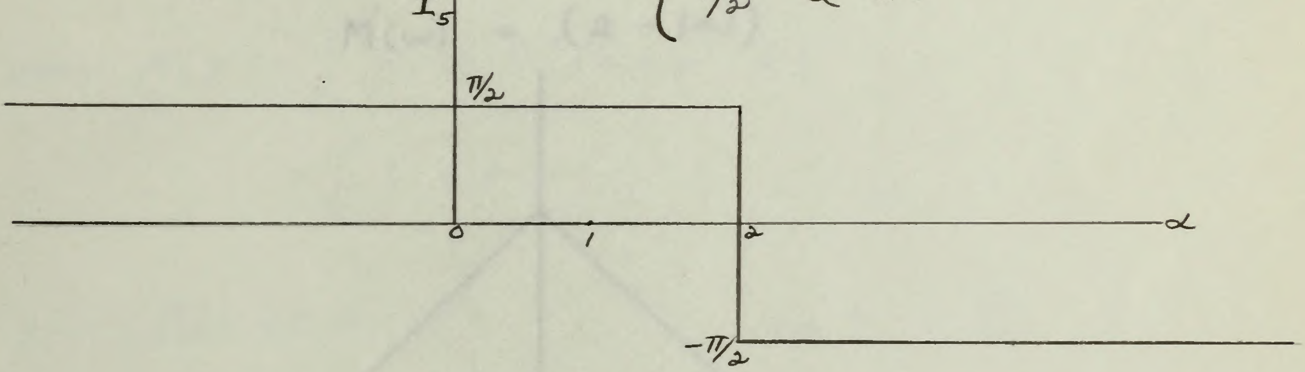
$$= -\int_0^\infty \frac{\sin(\alpha-2)\eta}{\eta}\,d\eta - \alpha\int_0^\infty \frac{\sin\eta\cos(1-\alpha)\eta}{\eta}\,d\eta$$

or

$$I_2 = I_5 + \alpha I_6$$

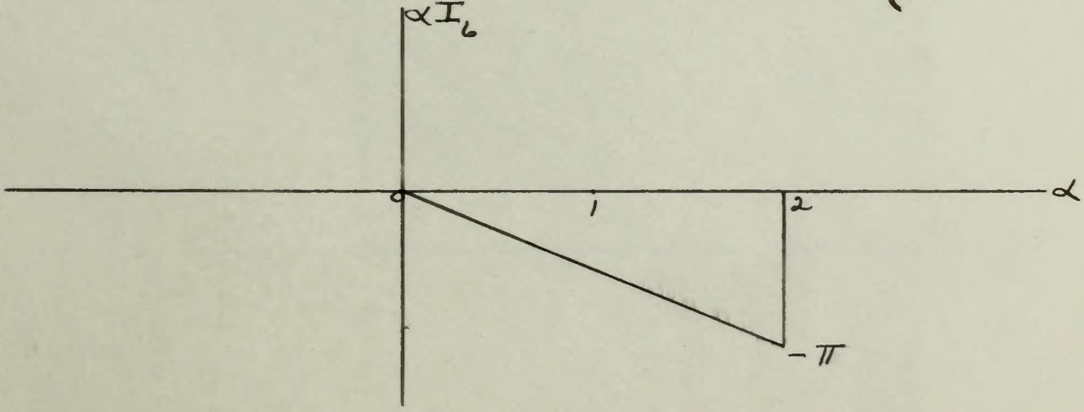
where:

$$I_5 = -\int_0^\infty \frac{\sin(\alpha-2)}{\eta}\,d\eta = \begin{cases} -\pi/2 & \alpha > 2 \\ 0 & \alpha = 2 \\ \pi/2 & \alpha < 2 \end{cases}$$



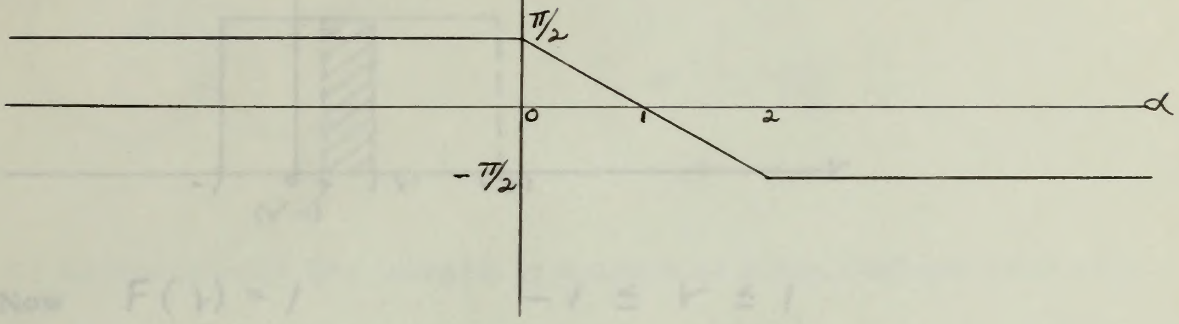
and

$$\alpha I_6 = -\alpha\int_0^\infty \frac{\sin\eta\cos(1-\alpha)\eta}{\eta}\,d\eta = -\alpha\begin{cases} 0 & \alpha < 0;\ \alpha > 2 \\ \pi/4 & \alpha = 0;\ \alpha = 2 \\ \pi/2 & 0 < \alpha < 2 \end{cases}$$



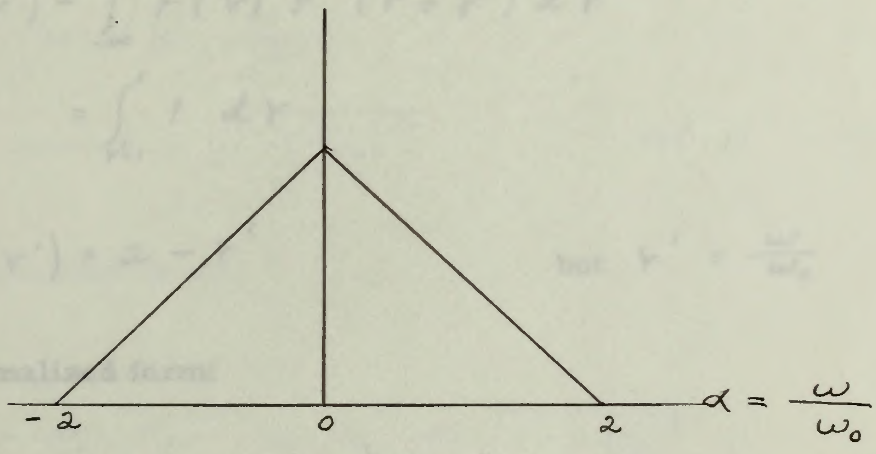
and the two add to give

$$I_2 = I_5 + \alpha I_6$$

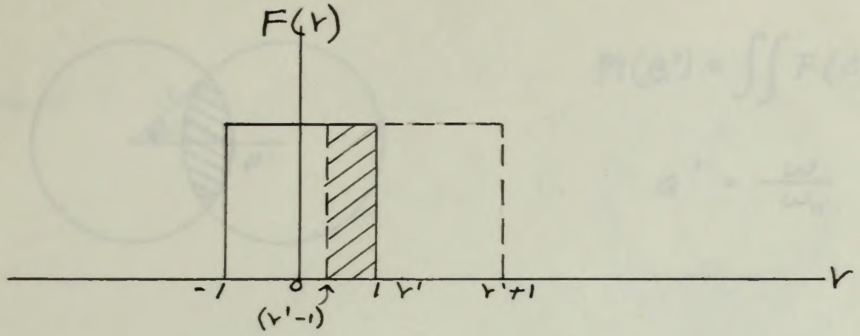


Now adding I_1 and I_2 graphically and normalizing:

$$M'(\omega) = (2 - |\alpha|)$$



APPENDIX 2-II



Now $F(r) = 1 \qquad -1 \leq r \leq 1$

$$= 0 \qquad |r| > 1$$

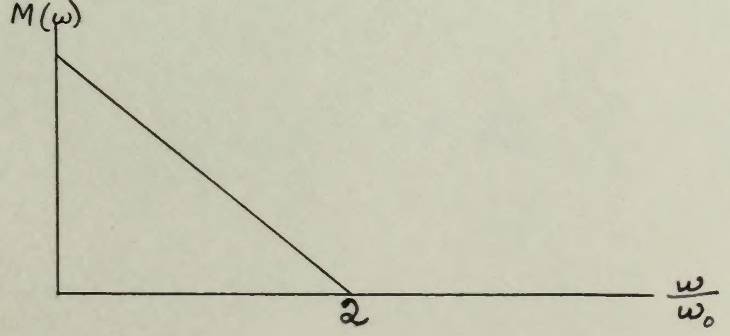
hence: $M(r') = \int_{-\infty}^{\infty} F(r)\, F^*(r + r')\, dr$

$$= \int_{r'-1}^{1} 1 \, dr$$

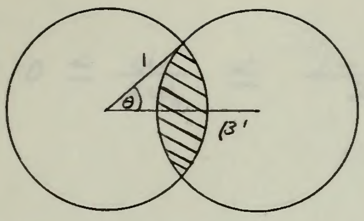
$$M(r') = 2 - r'$$ but $r' = \frac{\omega}{\omega_0}$

hence in normalized form:

$$M(\omega) = \left(1 - \frac{\omega}{2\omega_0}\right)$$



APPENDIX 2-III



$$M(\beta') = \iint F(\beta, \gamma)\, F^{*}(\beta+\beta', \gamma)\, d\beta\, d\gamma$$

$$\beta' = \frac{\omega}{\omega_0}$$

Geometrically this integral is a measure of the common area as a function of the shift between centers.

i.e. $$A = 2(\theta - \sin\theta \cos\theta)$$

$$\cos\theta = \frac{\beta'}{2}$$

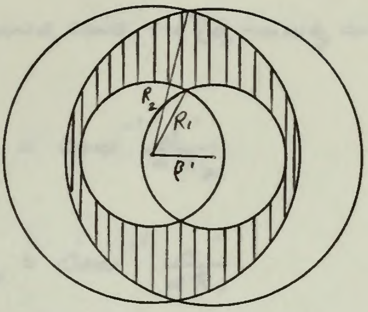
$$\sin\theta = \sqrt{1-(\beta'/2)^2}$$ and $$\beta' = \frac{\omega}{\omega_0}$$

Hence in normalized form:

$$M(\omega) = \frac{2}{\pi}\left[\cos^{-1}\frac{\omega}{2\omega_0} - \frac{\omega}{2\omega_0}\sqrt{1-\left(\frac{\omega}{2\omega_0}\right)^2}\,\right]$$

APPENDIX 2-IV

a) $0 \leq \frac{\omega}{2\omega_0} \leq \frac{1-\alpha}{2}$



Again measuring the overlapping area as a function of the shift

$$A = 2R_2^2\left[\theta_1 - \sin\theta_1 \cos\theta_1 - \alpha^2\pi + \alpha^2(\theta_4 - \sin\theta_4 \cos\theta_4)\right]$$

where

$$\theta_1 = \cos^{-1}\frac{\beta'}{2R_2} \qquad \theta_4 = \cos^{-1}\frac{\beta'}{2R_1}$$

$$\theta_1 = \cos^{-1}\frac{\omega}{2\omega_0} \qquad \theta_4 = \cos^{-1}\frac{\omega}{2\alpha\omega_0}$$

or

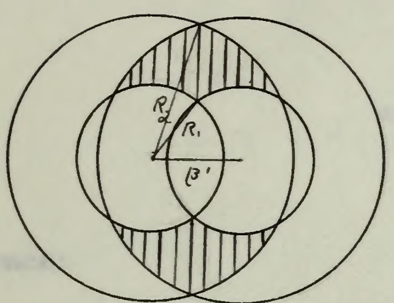
$$M(\omega) = \frac{2}{\pi}\left[\cos^{-1}\frac{\omega}{2\omega_0} - \frac{\omega}{2\omega_0}\sqrt{1-\left(\frac{\omega}{2\omega_0}\right)^2} + \alpha^2\left\{\cos^{-1}\frac{\omega}{2\alpha\omega_0} - \frac{\omega}{2\alpha\omega_0}\sqrt{1-\left(\frac{\omega}{2\alpha\omega_0}\right)^2}\right\} - \pi\alpha^2\right]$$

$$0 \leq \frac{\omega}{2\omega_0} \leq \frac{1-\alpha}{2}$$

b) $\frac{1-\alpha}{2} < \frac{\omega}{2\omega_0} < \frac{1+\alpha}{2}$

$$A = 2R_2^2\left[\theta_1 - \sin\theta_1\cos\theta_1 + \alpha^2(\theta_4 - \sin\theta_4\cos\theta_4) - \alpha^2\theta_3 - \theta_2 + \frac{\beta'}{R_2}\sin\theta_2\right]$$

where $\theta_1 = \cos^{-1}\frac{\beta'}{2R_2}$

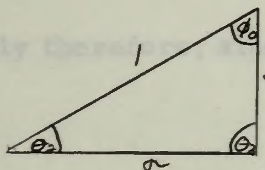


$$\theta_4 = \cos^{-1}\frac{\beta'}{2R_1}$$

$$\theta_3 = \cos^{-1}\frac{R_1^2 + \beta'^2 - R_2^2}{2\beta' R_1}$$

$$\theta_2 = \cos^{-1}\frac{R_2^2 + \beta'^2 - R_1^2}{2\beta' R_2}$$

That this expression is equivalent to equa. 2.134 can be shown from the following scheme where $\sigma = \frac{\beta'}{R_2}$



$$\frac{\sigma}{\sin\phi_0} = \frac{\alpha}{\sin\theta_2}$$

$$\phi_0 + \theta_2 + \theta_3 = \pi$$

$$\sigma\sin\theta_2 - \alpha^2\theta_3 - \theta_2 = \alpha\sin\phi_0 + (\phi_0 + \theta_2 - \pi)\alpha^2 - \theta_2$$

$$= -\pi\alpha^2 + \frac{\phi_0}{2} + \frac{\alpha^2\phi_0}{2} + \alpha\sin\phi_0 - \frac{\pi}{2} + \frac{C}{2} + \frac{\alpha^2\pi}{2} - \frac{\alpha^2 C}{2}$$

$$= -\pi\alpha^2 + \frac{\phi_0}{2} + \frac{\alpha^2\phi_0}{2} + \alpha\sin\phi_0 - (1-\alpha^2)\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

where

$$C = \pi - \phi_0 - 2\theta_2$$

b) cont.

Now $\tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\frac{C}{2} = \frac{1+\alpha}{1-\alpha} \tan\frac{1}{2}[A - B]$

where $A = \phi_0 + \theta_2$

$B = \theta_2$

and $A + B + C = \pi$

Hence:

$$\frac{\pi}{2} - \frac{C}{2} = \tan^{-1} \frac{1+\alpha}{1-\alpha} \tan\frac{\phi_0}{2}$$

thus

$$\sigma \sin\theta_2 - \alpha^2\theta_3 - \theta_2 = (1+\alpha^2)\frac{\phi_0}{2} + \alpha\sin\phi_0 - \pi\alpha^2 - (1-\alpha^2)\tan^{-1}\left(\frac{1+\alpha}{1-\alpha}\tan\frac{\phi_0}{2}\right)$$

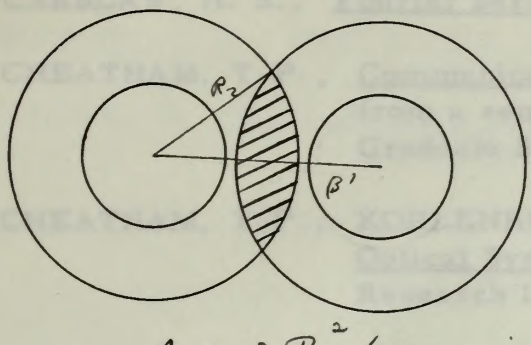
Finally therefore; since $\sigma = \frac{\beta'}{R_2} = \frac{\omega}{\omega_0}$

$$M(\omega) = \frac{2}{\pi}\Big[\cos^{-1}\frac{\omega}{2\omega_0} - \frac{\omega}{2\omega_0}\sqrt{1-\left(\frac{\omega}{2\omega_0}\right)^2} + \alpha^2 (\cos^{-1}\frac{\omega}{2\alpha\omega_0} - \frac{\omega}{2\alpha\omega_0}\sqrt{1-\left(\frac{\omega}{2\alpha\omega_0}\right)^2}$$

$$- \pi\alpha^2 + (1+\alpha^2)\frac{\phi_0}{2} + \alpha\sin\phi_0 - (1-\alpha^2)\tan^{-1}\left(\frac{1+\alpha}{1-\alpha}\tan\frac{\phi_0}{2}\right)\Big]$$

$$\frac{1-\alpha}{2} \le \frac{\sigma}{2} \le \frac{1+\alpha}{2}$$

c) $\frac{1+\alpha}{2} \le \frac{\omega}{2\omega_0} \le 1$



$$A = 2R_2^2(\theta_1 - \sin\theta_1 \cos\theta_1)$$

so that

$$M(\omega) = \frac{2}{\pi}\left[\cos^{-1}\frac{\omega}{2\omega_0} - \frac{\omega}{2\omega_0}\sqrt{1-\left(\frac{\omega}{2\omega_0}\right)^2}\,\right]$$

$$\frac{1+\alpha}{2} \le \frac{\omega}{2\omega_0} \le 1$$

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ABSTRACT

It is the purpose of this paper to present a method for dealing with optical processes in terms of their capability for preserving object detail in the final image. In general, the main body of the dissertation is divided into three sections. Section I presents, in detailed fashion, a comprehensive background for the subsequent use of Fourier series and integrals. As a starting point, the Sturm-Liouville differential equation is taken and the orthogonality principle for eigenfunctions is developed from it. Applying the results to the one dimensional wave equation, the Fourier series representation is derived along with the expression for discrete eigenvalues. By extending the finite boundaries to infinity, the Fourier integral representation is presented along with the expression for continuous eigenvalues of periodic and transient functions respectively is then presented. For problems possessing rotational symmetry, the Fourier Bessel series is developed from the wave equation in polar coordinates. A demonstration of the orthogonality condition for Bessel functions over an infinite range is presented. The section closes with a brief description of the conditions for performing a Fourier transformation and the interpretation of a linear passive system.

Section II deals exclusively with the application of linear network theory to several problems arising in diffraction theory. Emphasis is placed on the methods of evaluating and the interpretation of the modulation function as a contrast rendition versus spatial frequency curve for sine wave targets. Starting with the wave equation for one of the components of the electro-magnetic field quantities, the resultant Kirchhoff-Huygens formulation of the diffraction

problem is derived making use of the Green's function. With the usual feasible approximations made in the optical case, it is shown that the amplitude of the disturbance in the image plane is the two dimensional Fourier transform of the phase function over the aperture, a result said to have first been pointed out by Michelson. The phase function is such that for an aberrationless system, the actual wave front coincides with the reference sphere and hence the phase function is unity over the aperture and zero elsewhere.

With the added relation that the modulation function and intensity distribution for a point source are Fourier transform pairs, it is then shown that the modulation function can be evaluated by means of a convolution integral of the phase function with its complex conjugate. This latter relationship was pointed out by Luneberg and made use of recently by Marechal. The various Fourier relationships are then demonstrated for the one dimensional problem of diffraction by a single slit. The interpretation of the modulation function in terms of contrast rendition and resolution limit is then emphasized for periodic targets. In the two dimensional problem of the circular aperture the geometrical scheme and Steel's method for computing the modulation function are demonstrated. However, since the modulation function is defined as the Fourier transform of the point image intensity distribution, a method for taking the Fourier-Bessel transform of the intensity distribution for a circular aperture is presented. The method used is that of integration with respect to a parameter, the results of which are in complete accord with the other methods of determining the modulation function. Finally the same method of integration is applied to the problem of the annulus and a family of modulation curves with the ratio

of inner to outer radii as a parameter is plotted.

In section III, the Fourier analysis of several simple scanning systems and an ideal defocused lens are presented. Included in the analysis is a discussion of the phenomena of spurious resolution for periodic targets. A general treatment of edge deterioration and image quality is then given for any symmetrical, degrading point image intensity distributions.

Attention is then focused on the problem of synthesizing an optical system as opposed to analysis previously treated. Before such a procedure is outlined a general treatment of random functions is presented and the need for space averages of the functions involved is shown. This discussion leads into the concepts of correlation functions and power density spectra as Fourier transform pairs. In the light of the fact that most optical systems employ incoherent illumination, various restrictions and limitations on the design of an optimum linear filter in the Wiener-Hopf sense are presented. A brief description of specific synthesizing procedures actually in existence is then listed. Finally, recommendations for future research in this field are outlined along with the summary of the dissertation proper.

AUTOBIOGRAPHY

Born in Boston, Mass., November 29, 1927, the son of Mr. and Mrs. Frederick T. O'Neill, the author received his high school diploma from the Boston Public Latin School in June 1945. In the fall of that year he enrolled at Boston College and received his Bachelor of Arts degree cum laude in June 1949 having been awarded a full scholarship in his senior year at that institution.

For the academic year (1949-1950), the author attended Tufts Graduate School under a teaching fellowship in the Physics department. In the fall of 1950, he enrolled at Boston University as a candidate for the degree of Doctor of Philosophy in Physics. During the summer of 1951, he was awarded a Master of Arts degree from that institution and was married to Marian Christine Kohr of Needham, Mass.

In February 1951, the author accepted a part time position as technical engineer with the Gas Turbine Division of the General Electric Co. In February 1952, he was appointed Research Assistant at the Boston University Physical Research Laboratory. During the Intersession of 1953, the author taught the Theoretical Physics course at Boston University.

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